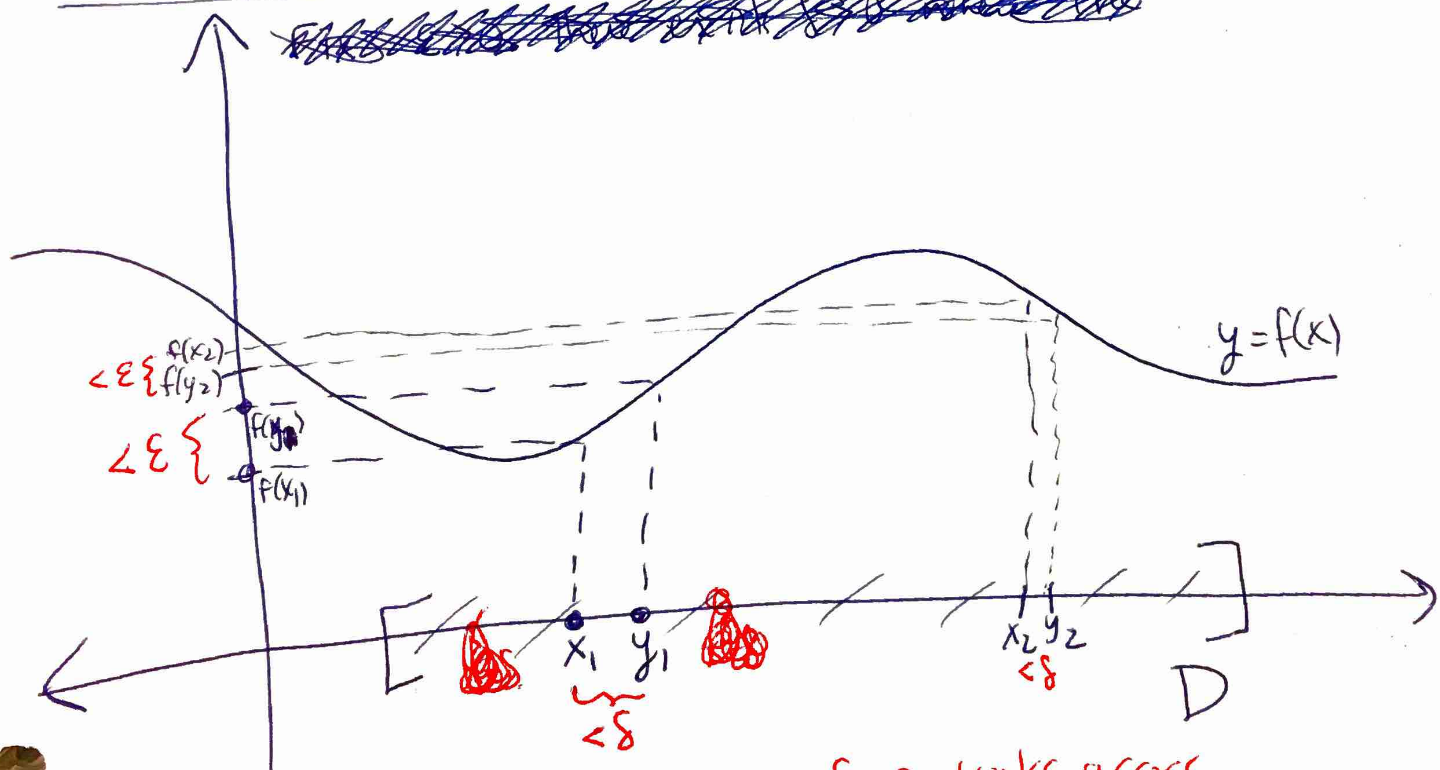
# Uniform Continuity

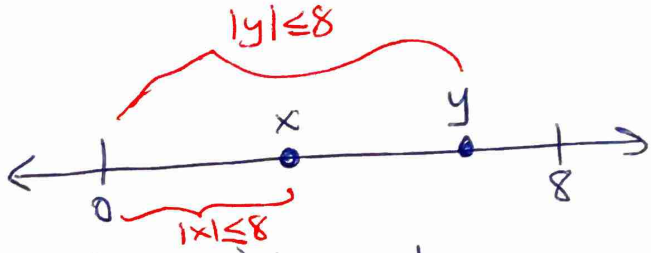
**Def:** Let $f : D \to \mathbb{R}$ be a function where $D \subseteq \mathbb{R}$. We say that $f$ is uniformly continuous on $D$ if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for every $x, y \in D$ we have that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.




If $\varepsilon > 0$, then the same $\delta > 0$ works across all of $D$. Normal continuity is a local "pointwise" property.

**Ex:** Let's show that $f(x)=x^2$ is uniformly continuous on $[0,8]$.

Let $\varepsilon > 0$ be fixed.
Suppose $x, y \in [0,8]$.




Then,

$$|f(x)-f(y)| = |x^2-y^2| = |x+y||x-y| \le (|x|+|y|)|x-y|$$
$$\le (8+8)|x-y| = 16|x-y|.$$

Let $\delta = \frac{\varepsilon}{16}$.

Then if $x,y \in [0,8]$ and $|x-y| < \delta$, then

$$|f(x)-f(y)| \le 16|x-y| < 16 \cdot \delta = 16 \cdot \frac{\varepsilon}{16} = \varepsilon.$$

$\blacksquare$

~~Prove that uniform continuity implies continuity.~~

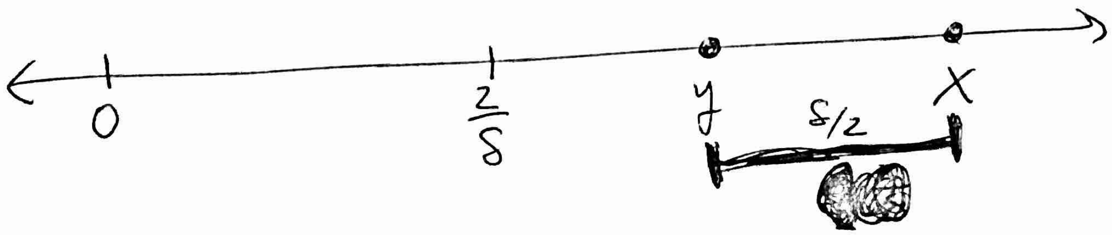
Note: $f(x)=x^2$ is continuous on all of $\mathbb{R}$. However, 62'

Ex: $f(x)=x^2$ is not uniformly continuous on $[0,\infty)$.

pf: Let $\varepsilon=1$.

We want to show that if $\delta>0$ then there exist $x,y\in[0,\infty)$ such that $|x-y|<\delta$ but $|x^2-y^2|>\varepsilon$.

Suppose $\delta>0$.
Pick $x,y\in\mathbb{R}$ such that $|x-y|=\frac{\delta}{2}$ and
$x,y>\frac{2}{\delta}>0$.




Then,

$$|x^2-y^2|=|x-y||x+y|=\left(\frac{\delta}{2}\right)(x+y)>\frac{\delta}{2}\left(\frac{2}{\delta}+\frac{2}{\delta}\right)=2>\varepsilon.$$

$\blacksquare$

**Ex:** Let $f(x)=\sqrt{x}$.

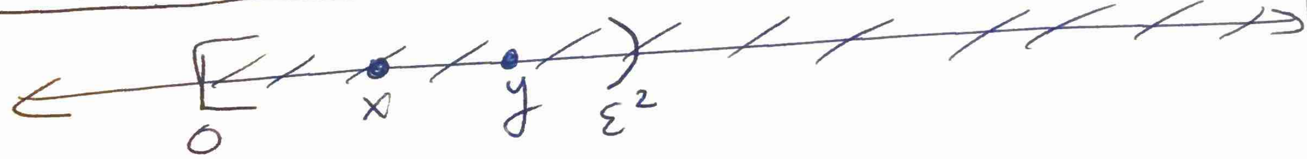
We show that $f$ is uniformly continuous on $[0,\infty)$.

Let $\varepsilon>0$ be fixed, choose $\delta=\varepsilon^2$.

Suppose that $x,y\in[0,\infty)$, with $|x-y|<\delta$.

~~But either both of x and y are in $[0,\varepsilon^2)$~~

**case 1:** Both of $x$ & $y$ are in $[0,\varepsilon^2)$.



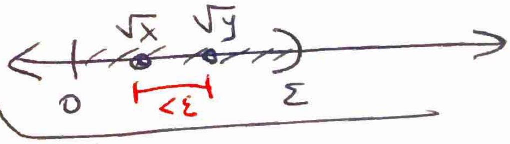

~~Thus, $\sqrt{x}$ and $\sqrt{y}$~~

So, $0\le x<\varepsilon^2$ and $0\le y<\varepsilon^2$.

Thus, $0\le\sqrt{x}<\varepsilon$ and $0\le\sqrt{y}<\varepsilon$.

Using the fact that $\sqrt{x}$ is an increasing function

So, $|f(x)-f(y)|=|\sqrt{x}-\sqrt{y}|<\varepsilon$




**case 2:** One of $x$ & $y$ (or both?) lies outside of $[0,\varepsilon^2)$.

So either $\varepsilon^2\le x$ or $\varepsilon^2\le y$. So either $\sqrt{\varepsilon^2}\le\sqrt{x}$ or $\sqrt{\varepsilon^2}\le\sqrt{y}$.

So, $\sqrt{x}+\sqrt{y}\ge\sqrt{\varepsilon^2}=\varepsilon$. Thus, $|f(x)-f(y)|=\left|\sqrt{x}-\sqrt{y}\cdot\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right|$

$=\left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right|=\frac{|x-y|}{\sqrt{x}+\sqrt{y}}\le\frac{|x-y|}{\varepsilon}<\frac{\delta}{\varepsilon}=\varepsilon.$

$(\sqrt{x})^2=x$ if $x\ge0$

**Conclusion:** In either case, if $|x-y|<\delta$, then $|f(x)-f(y)|<\varepsilon$.

**Def:** Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$.
If $f$ is uniformly continuous on $D$, then $f$ is continuous on $D$.

**pf:** Let $\varepsilon > 0$.
Since $f$ is uniformly continuous on $D$, there exists $\delta > 0$ so that if $|x-y| < \delta$ and $x, y \in D$, then $|f(x) - f(y)| < \varepsilon$.
Let $a \in D$.
Then if $x \in D$ and $0 < |x-a| < \delta$, then

$$|f(x) - f(a)| < \varepsilon.$$

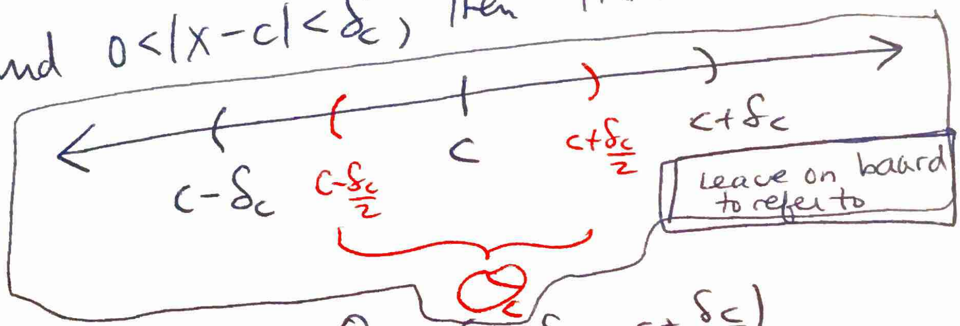
So $f$ is continuous at $a$. $\blacksquare$

**Theorem:** Let $D \subseteq \mathbb{R}$. Suppose that $D$ is closed and bounded (compact). and $f: D \to \mathbb{R}$ is a continuous function on all of $D$.
Then $f$ is uniformly continuous on $D$.

**proof:**

Let $\varepsilon > 0$ be fixed. Since $f$ is continuous at each $c \in D$ we can find $\delta_c > 0$, depending on $c$, such that

if $x \in D$ and $0 < |x - c| < \delta_c$, then $|f(x) - f(c)| < \frac{\varepsilon}{2}$.




For each $c \in D$, let $O_c = \left(c - \frac{\delta_c}{2}, c + \frac{\delta_c}{2}\right)$.

Then $X = \{ O_c \mid c \in D \}$ is an open covering of $D$.

Since $D$ is compact, there exit $c_1, c_2, \dots, c_m \in D$

$X' = \{ O_{c_1}, O_{c_2}, \dots, O_{c_m} \}$ is an open cover of $D$.

where

Suppose that $x \in \Theta_{c_k}$ for some $k$. Then, $|x-c_k| < \frac{\delta_{c_k}}{2} < \delta_{c_k}$.

So, $|f(x)-f(c_k)| < \frac{\varepsilon}{2}$.

Let $\delta = \min\left\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \ldots, \frac{\delta_m}{2}\right\}$.

Suppose that $x, y \in D$ with $|x-y| < \delta$.

We now show that $|f(x)-f(y)| < \varepsilon$.

Since $\{\Theta_{c_1}, \ldots, \Theta_{c_m}\}$ is an open cover of $D$ and $x \in D$ we have that $x \in \Theta_{c_k}$ for some $k$.

So, $|f(x)-f(c_k)| < \frac{\varepsilon}{2}$.

Also,

$$|y-c_k| = |(y-x)+(x-c_k)| \le |y-x| + |x-c_k|$$
$$< \delta + \frac{\delta_{c_k}}{2} \le \frac{\delta_{c_k}}{2} + \frac{\delta_{c_k}}{2} = \delta_{c_k}$$

Thus, $|y-c_k| < \delta_{c_k}$.

So, $|f(y)-f(c_k)| < \frac{\varepsilon}{2}$.

Thus,
$$|f(x)-f(y)| = |f(x)-f(c_k)+f(c_k)-f(y)|$$
$$\le |f(x)-f(c_k)| + |f(c_k)-f(y)|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$