Math 4650

8/25/25

There's one more property about $\mathbb{R}$ that we will assume. It's called the completeness axiom. Let's work up to stating it.

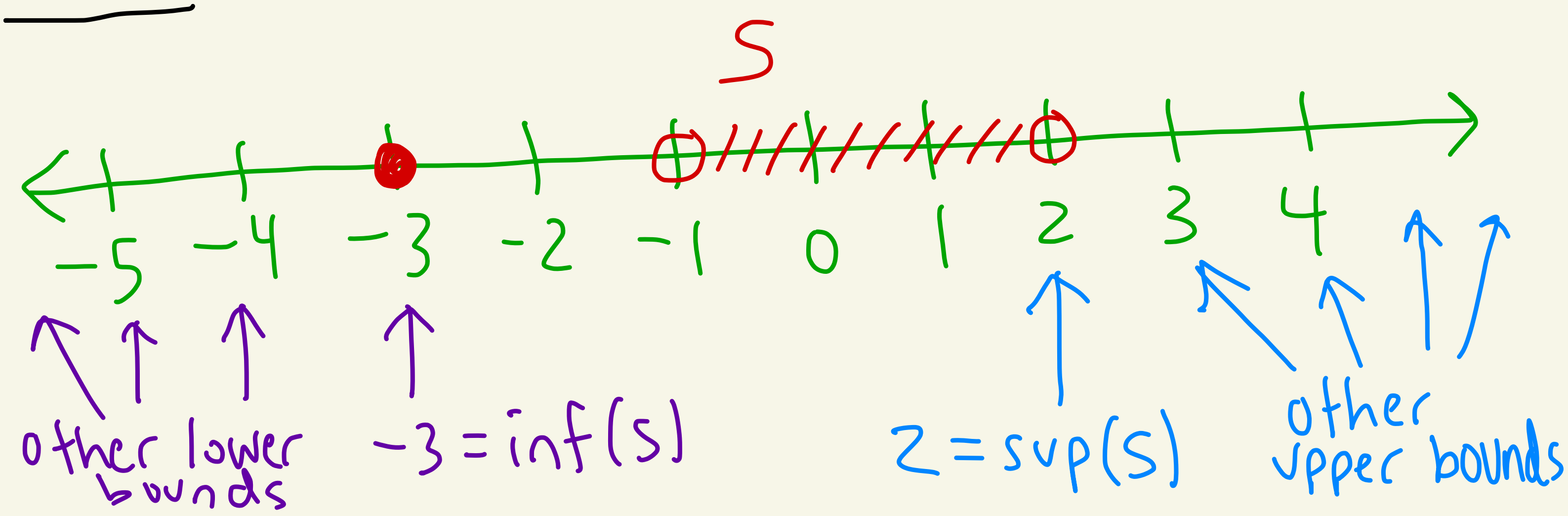
Def: Let $S \subseteq \mathbb{R}$ where $S \neq \emptyset$.

- We say that b is an <u>upper bound for S</u> if $x \leq b$ for all $x \in S$. If there exists an upper bound for S, then we say that S is <u>bounded from above</u>.
- If b is an upper bound for S and $b \leq c$ for all other upper bounds c of S, then we call b the <u>least upper bound for S</u>, or <u>supremum of S</u>, and we write $b = \sup(S)$.
- We say that b is a <u>lower bound for S</u> if $b \leq x$ for all $x \in S$. If there exists a lower bound

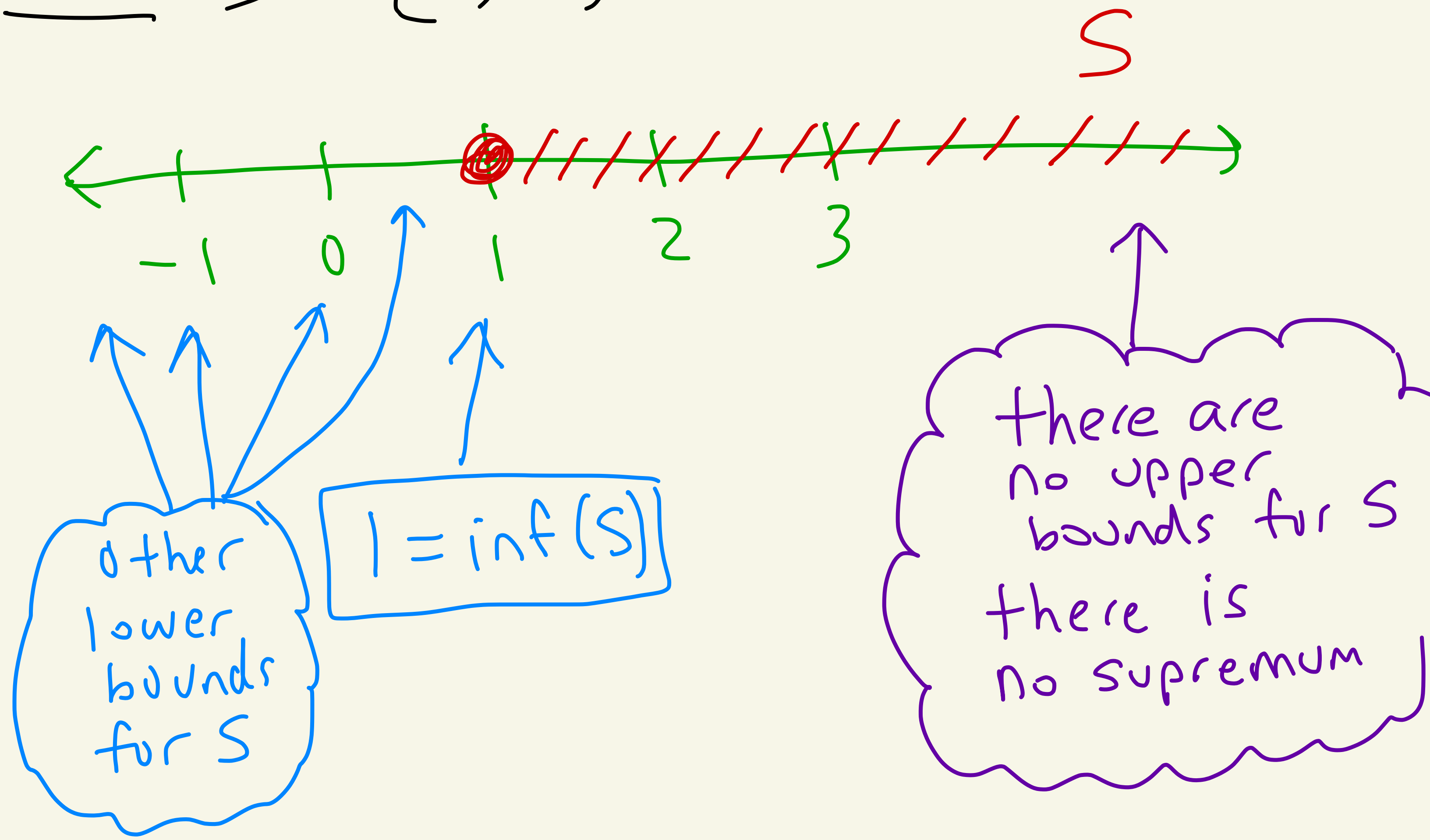
for S then we say that S is <u>bounded from below.</u>

- If b is a lower bound for S and $c \leq b$ for all other lower bounds c of S, then we call b the <u>greatest lower bound for S</u>, or <u>infimum of S</u>, and we write $b = \inf(S)$.

<u>Ex:</u> $S = (-1, 2) \cup \{-3\}$



Ex: $S = [1, \infty)$



Theorem: Let $S \subseteq \mathbb{R}$, $S \neq \emptyset$.
If $\sup(S)$ exists, then its unique.
If $\inf(S)$ exists, then its unique.

proof: HW 1 $\blacksquare$

The completeness axiom for $\mathbb{R}$

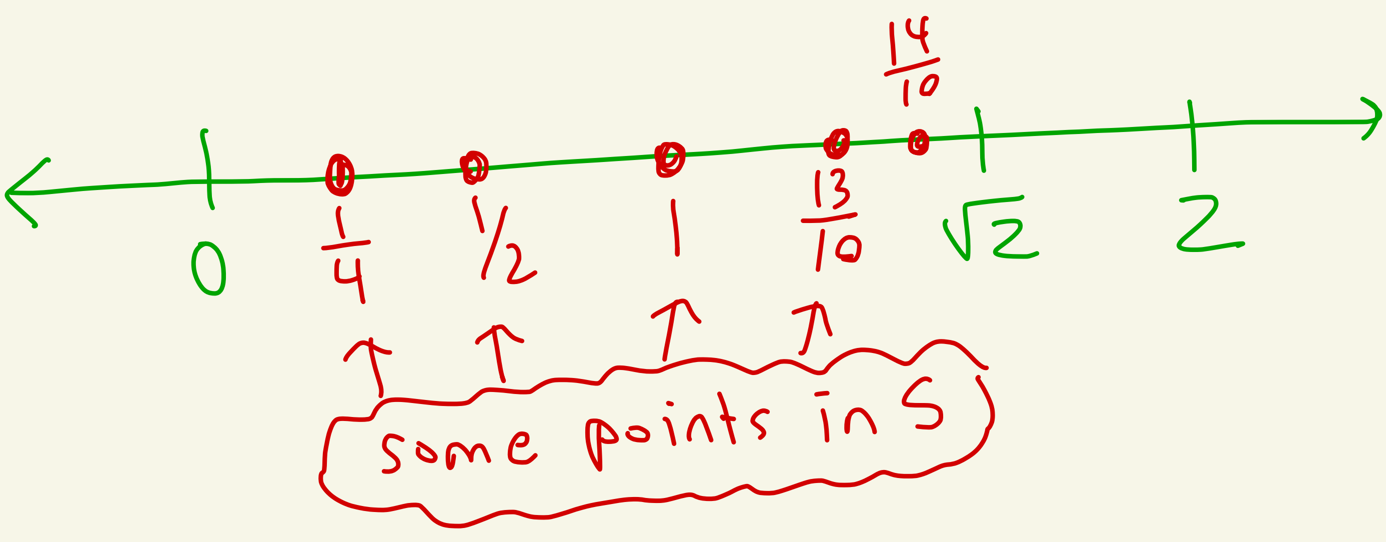
Let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.

If S is bounded from above, then $\sup(S)$ exists in $\mathbb{R}$.

If S is bounded from below, then $\inf(S)$ exists in $\mathbb{R}$

Note: $\mathbb{R}$ has this property, but $\mathbb{Q}$ does not. Let

$$S = \{ x \mid x \in \mathbb{Q}, 0 < x, x^2 < 2 \}$$



S is bounded from above, by say 2, but sup(S) does not exist. The points in S get closer and closer to $\sqrt{2}$ but don't reach $\sqrt{2}$. The sup would be $\sqrt{2}$ but thats not in $\mathbb{Q}$.

<u>Theorem</u> (Archimedean property)
Let $x \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ with $x < n$

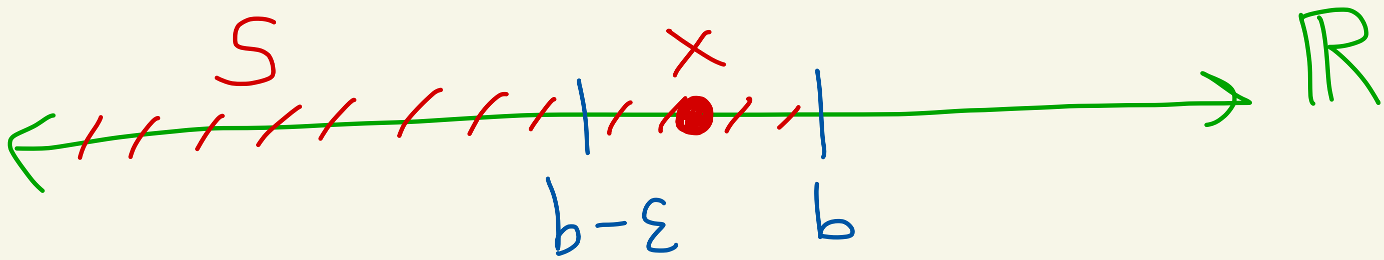
<u>Ex:</u> $x = 3\pi \approx 9.42\ldots$
$n = 10$

<u>proof:</u> See online notes. $\blacksquare$

Theorem (Inf-sup theorem)

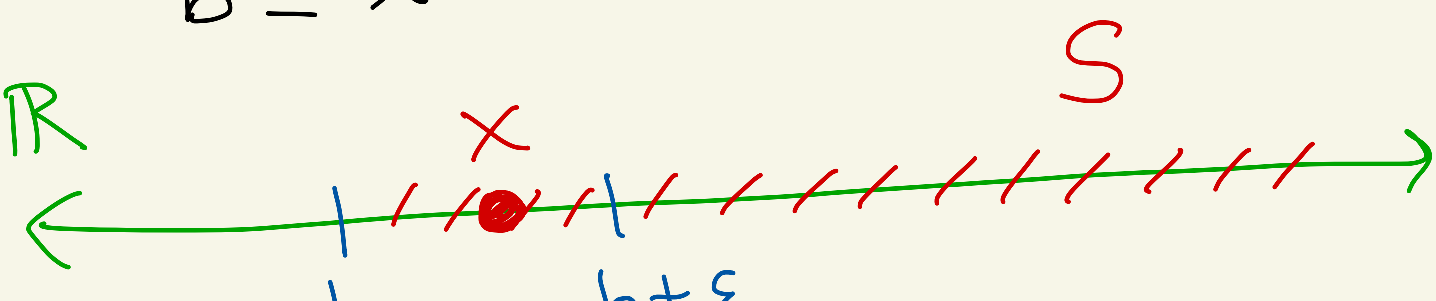
Let $S \subseteq \mathbb{R}$ where $S \neq \phi$.

(a) Suppose b is an upper bound for S. Then, b is the supremum of S if and only if for every $\varepsilon > 0$ there exists $x \in S$ satisfying $b - \varepsilon < x \leq b$.



(b) Suppose b is a lower bound for S. Then, b is the infimum of S if and only if for every $\varepsilon > 0$ there exists $x \in S$ satisfying

$b \leq x < b + \varepsilon$.



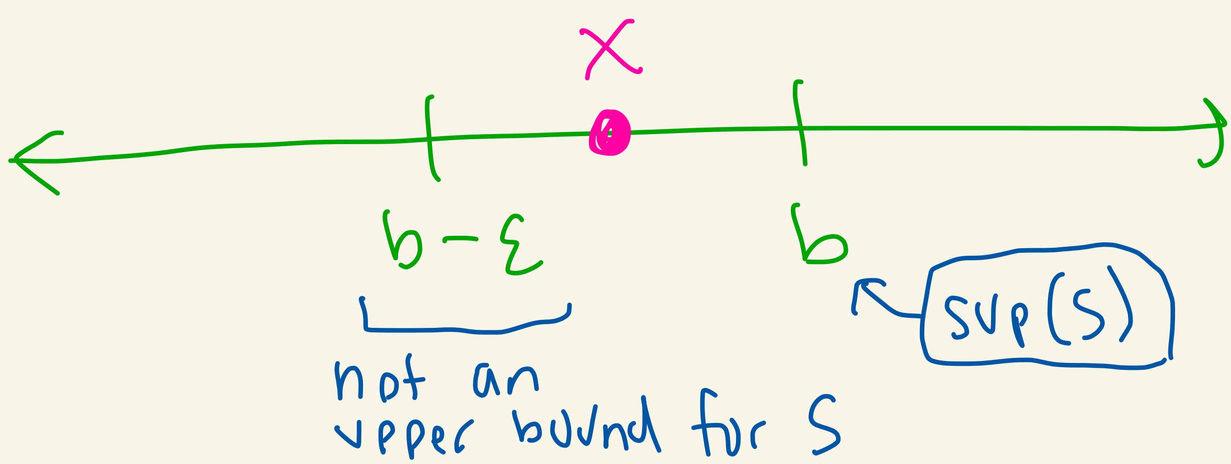
Proof: Let's prove (a).
Part (b) is similar.

Let b be an upper bound for S.

($\Rightarrow$) Assume b is the supremum for S.

Let $\varepsilon > 0$ be fixed.

Since $b - \varepsilon < b$ and b is the least upper bound for S we know that $b - \varepsilon$ is not an upper bound for S.



Since $b-\varepsilon$ is not an upper bound for S there exists $x \in S$ where $b-\varepsilon < x$.

Since $x \in S$ and $b = \sup(S)$ we know $x \leq b$.

Thus, $b-\varepsilon < x \leq b$.

($\Leftarrow$) Now suppose for every $\varepsilon > 0$ there exists $x \in S$ with $b-\varepsilon < x \leq b$.

We will show that b is the supremum of S.

We were given that b is an upper bound for S.

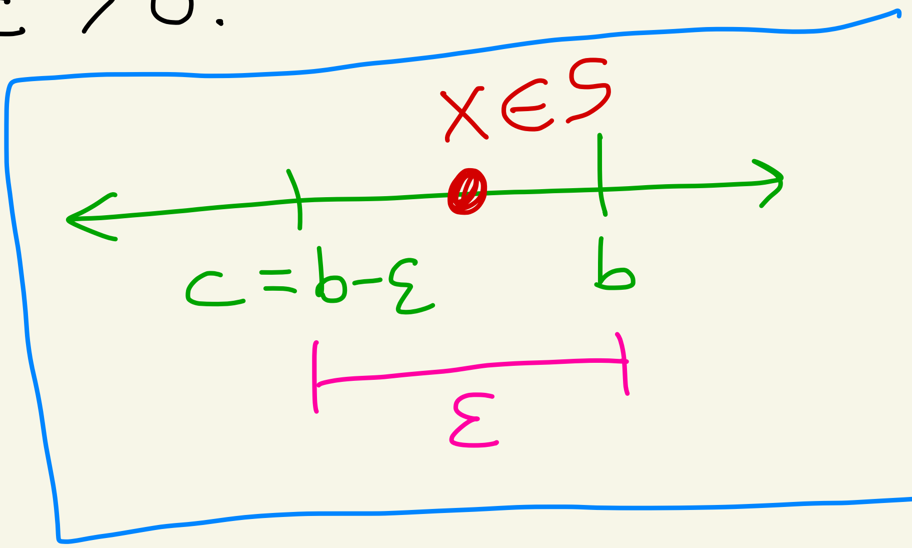
Let's show b is the least

upper bound for S.

Let $c < b$.

Let's show that c cannot be an upper bound for S.

Let $\varepsilon = b - c > 0$.



By assumption there exists $x \in S$ with $b - \varepsilon < x \leq b$.

So, $c < x \leq b$

Thus, c is not an upper bound for S.

So, b must be the least upper bound for S. $\blacksquare$