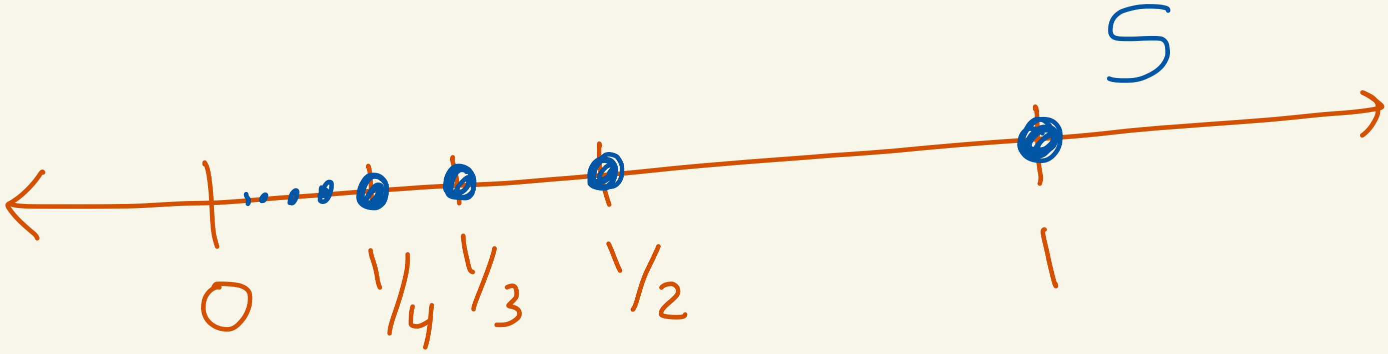
Math 4650
8/27/25

Ex: Let

$$S = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$$

$$= \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$$



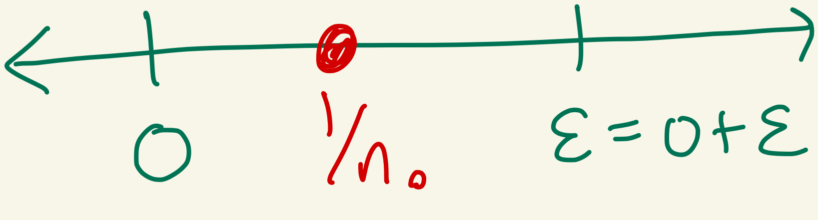
Let's show that $0 = \inf(S)$.

We know that 0 is a lower bound for S because

$$0 \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Let's use the Inf-Sup theorem.

Let $\varepsilon > 0$.



Pick some $n_0 \in \mathbb{N}$ where $n_0 > \frac{1}{\varepsilon}$

Then, $\frac{1}{n_0} < \varepsilon$

Then, $\frac{1}{n_0} \in S$ and $0 < \frac{1}{n_0} < 0 + \varepsilon$.

By the inf-sup theorem, $0 = \inf(S)$

▨

Def: Let $x \in \mathbb{R}$.

The **absolute value** of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|5| = 5$

$|-3| = -(-3) = 3$

Theorem:

Let $a, b, c \in \mathbb{R}$ with $c > 0$.

Then:

① $|ab| = |a| \cdot |b|$

② $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$

③ $|a| \leq c$ iff $-c \leq a \leq c$

④ $|a| < c$ iff $-c < a < c$

⑤ (triangle inequality)
$|a+b| \leq |a| + |b|$

⑥ $\big||a| - |b|\big| \leq |a-b|$

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proof: ①/② are in HW.

③

(⇒) Suppose $|a| \le c$.

If $a<0$, then $a < -a \overset{\text{def of abs. value}}{=} |a| \overset{\text{assumption}}{\le} c$.

If $a \ge 0$, then $-a \le a = |a| \le c$.

In both cases we get

$a \le c$ and $-a \le c$.

So, $-c \le a \le c$.

(⇐) Suppose $-c \le a \le c$.

Then, $-c \le a$ and $a \le c$.

So, $-a \le c$ and $a \le c$.

Thus, $|a| \le c$

Since $|a| = a$ or $|a| = -a$

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④ Similar proof to ③ proof.

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⑤ Note first that if $x \in \mathbb{R}$ then $|x| \leq |x|$.

Thus by taking $c = |x|$ and using part ③ we get

$$-|x| \leq x \leq |x|$$

③ says $|y| \leq c$ iff $-c \leq y \leq c$

Thus if $a, b \in \mathbb{R}$, then

$$-|a| \leq a \leq |a| \quad \text{and} \quad -|b| \leq b \leq |b|$$

Adding gives

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

Use part ③ again with $c = |a| + |b|$ to get:

$$|a + b| \leq |a| + |b|.$$

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⑥ HW ▨

Think of $|x-y|$ as the distance between x & y

We will use this alot:

Corollary:

Let $x, y, \varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.

Then:

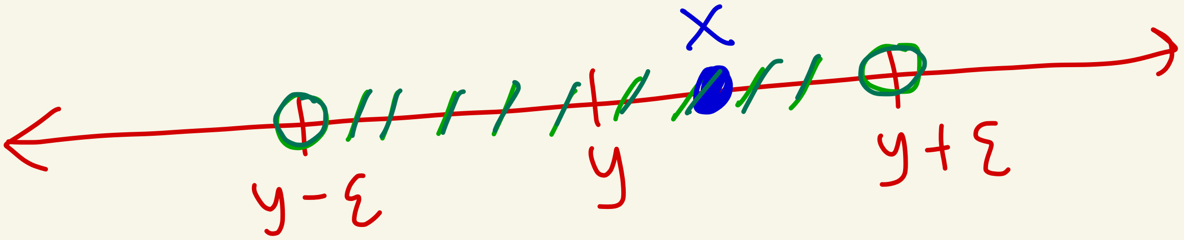
$$|x-y| < \varepsilon \quad \text{iff} \quad y-\varepsilon < x < y+\varepsilon$$

proof:

$$|x-y| < \varepsilon$$

$$\text{iff} \quad -\varepsilon < x-y < \varepsilon$$ part ③ above

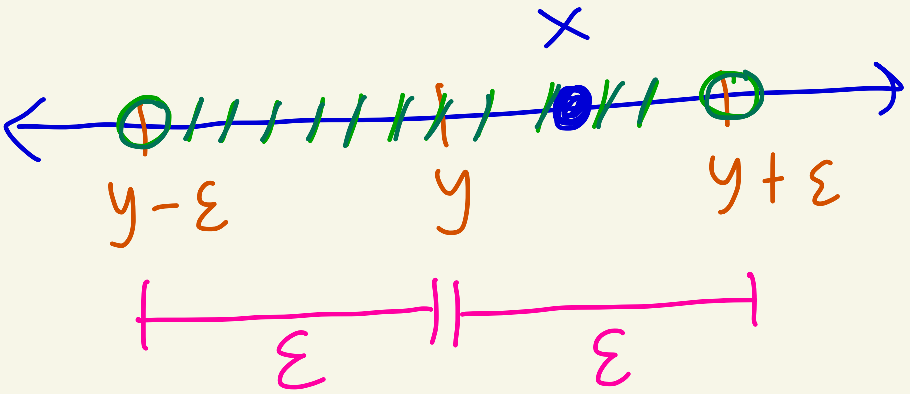
$$\text{iff} \quad y-\varepsilon < x < y+\varepsilon$$



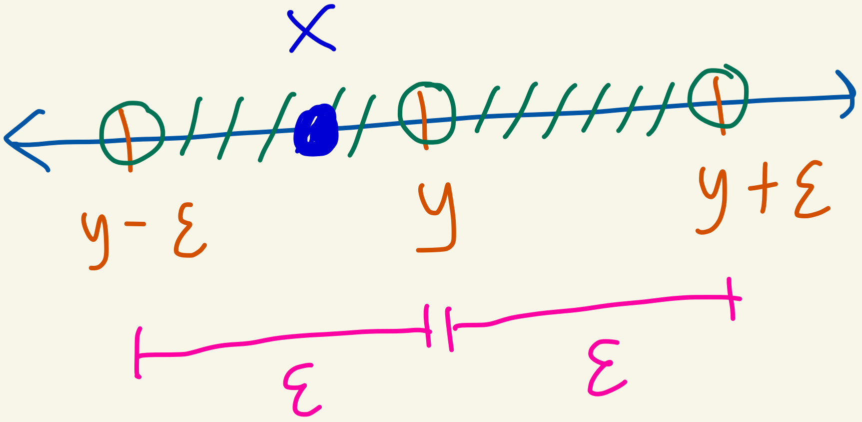
▨

Some pictures:

$$|x-y|<\varepsilon$$



$$\underbrace{0<|x-y|<\varepsilon}_{\text{forces } x\neq y}$$



Theorem: ($\mathbb{Q}$ is dense in $\mathbb{R}$)

Given $a, b \in \mathbb{R}$ with $a<b$, there exists $\frac{m}{n} \in \mathbb{Q}$ with $a < \frac{m}{n} < b$.

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proof:

By the Archimedean property there exists $n \in \mathbb{N}$ with $\frac{1}{b-a} < n$.

So, $\frac{1}{n} < b-a$.

Claim: There exists $m \in \mathbb{Z}$ with $m-1 \leq na < m$.

pf of claim:

Suppose $na > 0$,
Then by the Archimedean principle there exists a smallest

natural number k with $na < k$.

Ex: $na = 3.2$

possible k: $4, 5, 6, 7, 8, 9, 10, 11, \ldots$

smallest

Then, $k-1 \leq na < k$

Why?

What if $na < k-1$?

This would contract the choice of k.

In this case, set $m = k$.

Now suppose $na < 0$.

Let k be the smallest natural number with $-na \leq k$.

Then, $-k \leq na < -k+1$

Set $m = -k+1$.

Then, $m-1 \leq na < m$.

Claim

Since $na < m$ we get $a < \frac{m}{n}$.

Also,

$$m \le na + 1 < n\left(b - \frac{1}{n}\right)$$

$\boxed{m - 1 \le na}$

$\boxed{\frac{1}{n} < b - a \text{ implies } a < b - \frac{1}{n}}$

I will fix this proof.