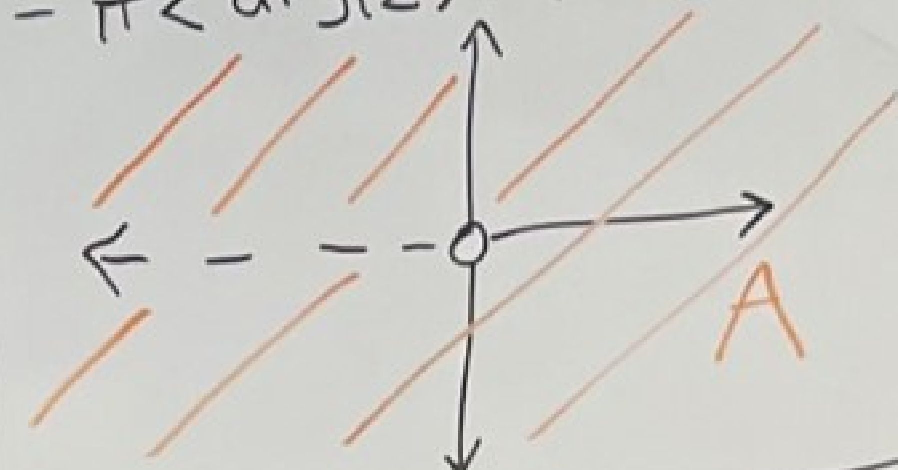
Ex: Let $b \in \mathbb{C}$. Let $f(z) = z^b = e^{b\log(z)}$

Suppose some branch of log is chosen and $A \subseteq \mathbb{C}$ is a set where that branch of log is analytic. $\longleftrightarrow$

Ex: principal branch

$\log(z) = \ln|z| + i\arg(z)$

$-\pi < \arg(z) < \pi$



Claim: f will be analytic on A and $f'(z) = bz^{b-1}$ when $z \in A$

proof: Let $z_0 \in A$.

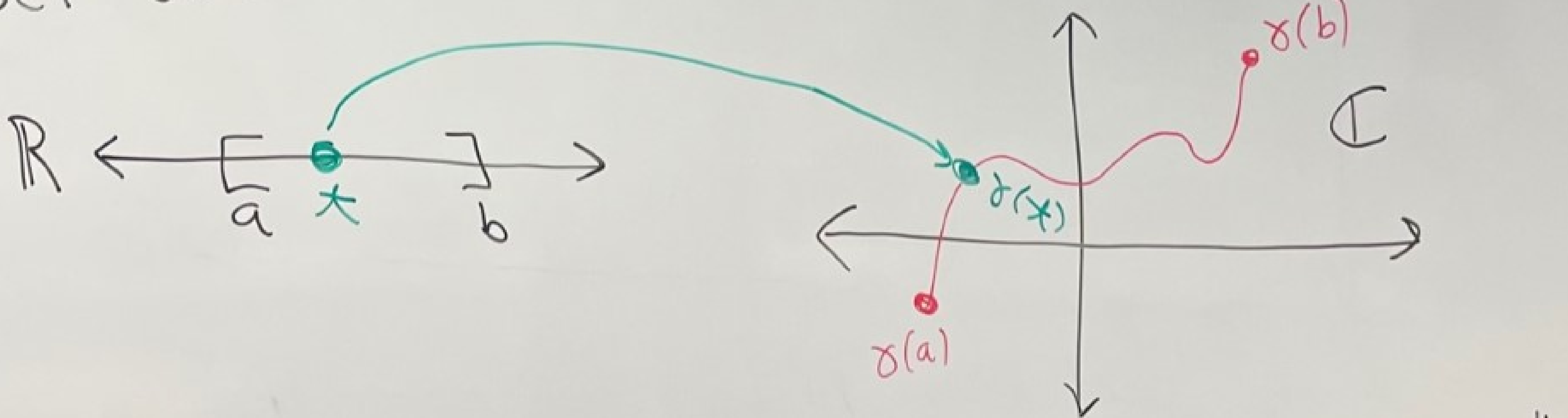
Then, log is differentiable at z_0 and since e^z is entire the composition $e^{b\log(z)}$ is differentiable at z_0.

And, $f'(z_0) = \left(e^{b\log(z_0)}\right) \cdot \frac{b}{z_0} = z_0^b \cdot \frac{b}{z_0} = bz_0^{b-1}$ $\blacksquare$

HW 6/7 – Integrals and path-connected

Def: Let $a, b \in \mathbb{R}$ and $a < b$. Let $\gamma : \underbrace{[a,b]}_{\text{interval in } \mathbb{R}} \to \mathbb{C}$

Set $\gamma(t) = u(t) + i\,v(t)$ where $u : [a,b] \to \mathbb{R}$ and $v : [a,b] \to \mathbb{R}$



- We say that γ is a <u>curve</u> (or <u>arc</u>) if u and v are continuous on $[a,b]$.
- If u' and v' exist on (a,b) then we define

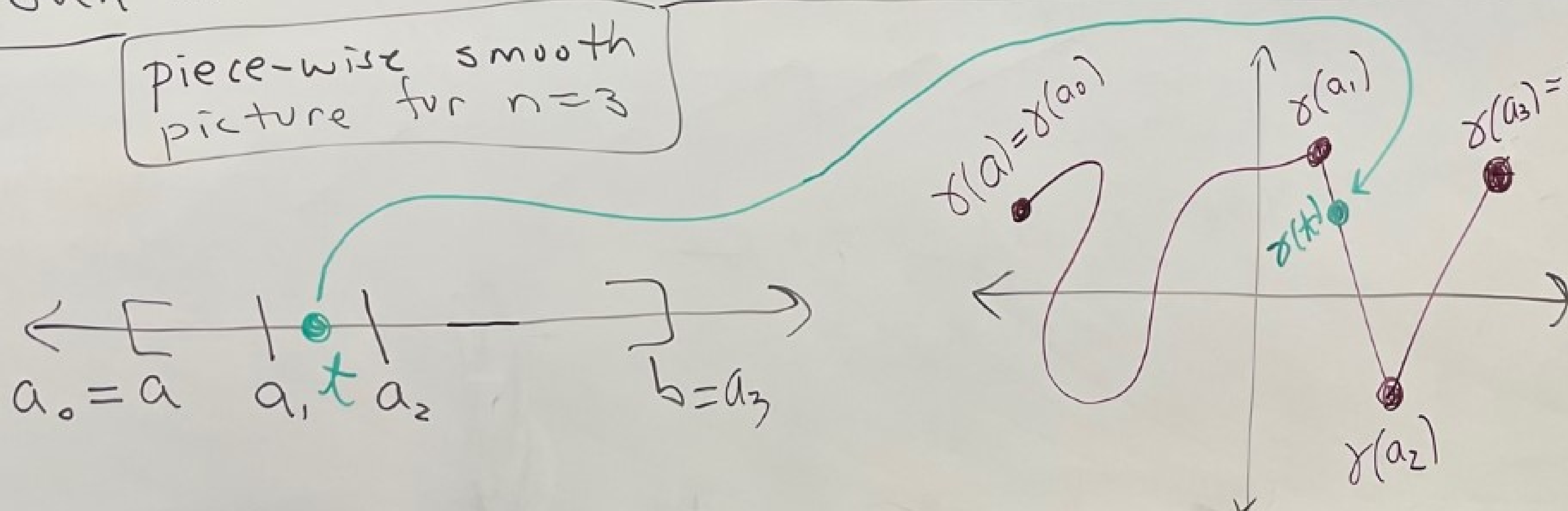
$$\gamma'(t) = u'(t) + i\,v'(t)$$

and we say γ' exists and γ is <u>differentiable</u>.

(def continued...)

- We say that γ is a <u>smooth curve</u> if γ is a curve, and γ is differentiable, and u' and v' are continuous on $[a,b]$.
- γ is called <u>piecewise-smooth</u> if we can divide the interval $[a,b]$ into finitely many sub-intervals $a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$ such that γ is smooth on each $[a_i, a_{i+1}]$

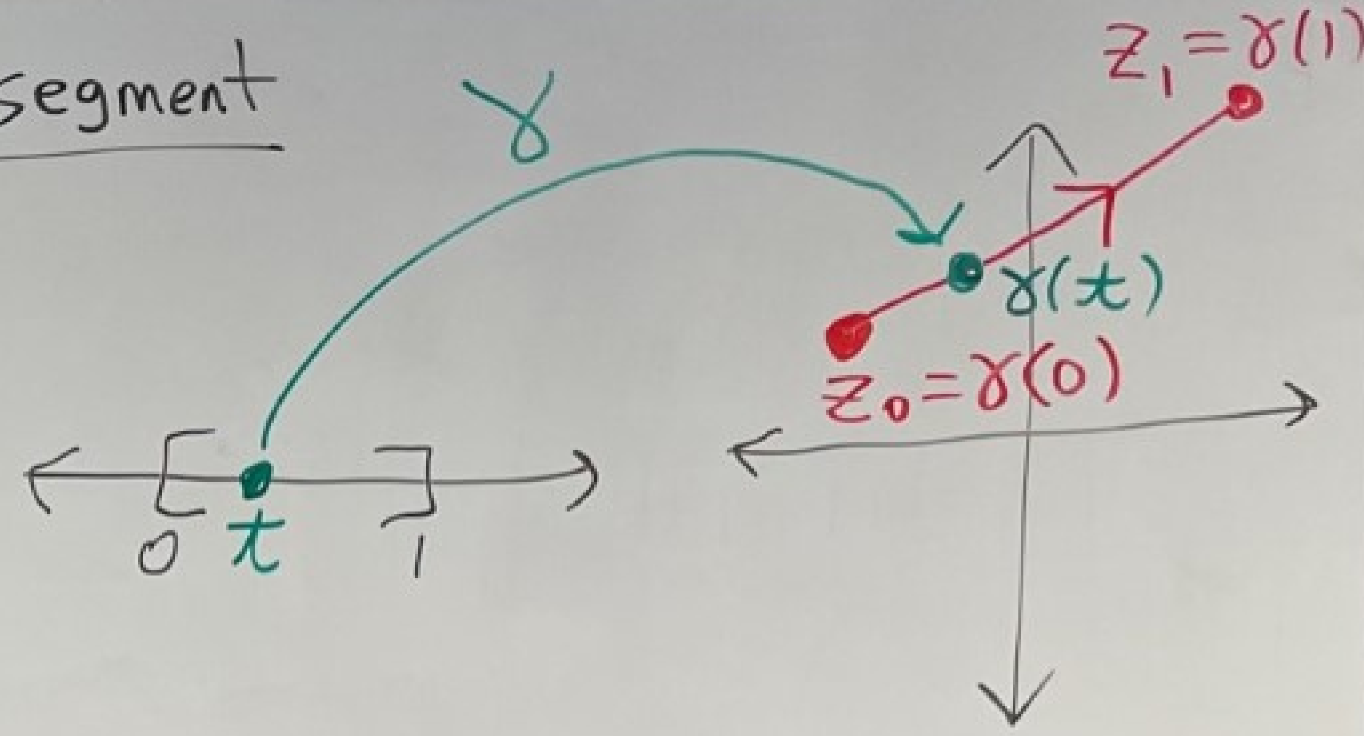
piece-wise smooth picture for $n=3$



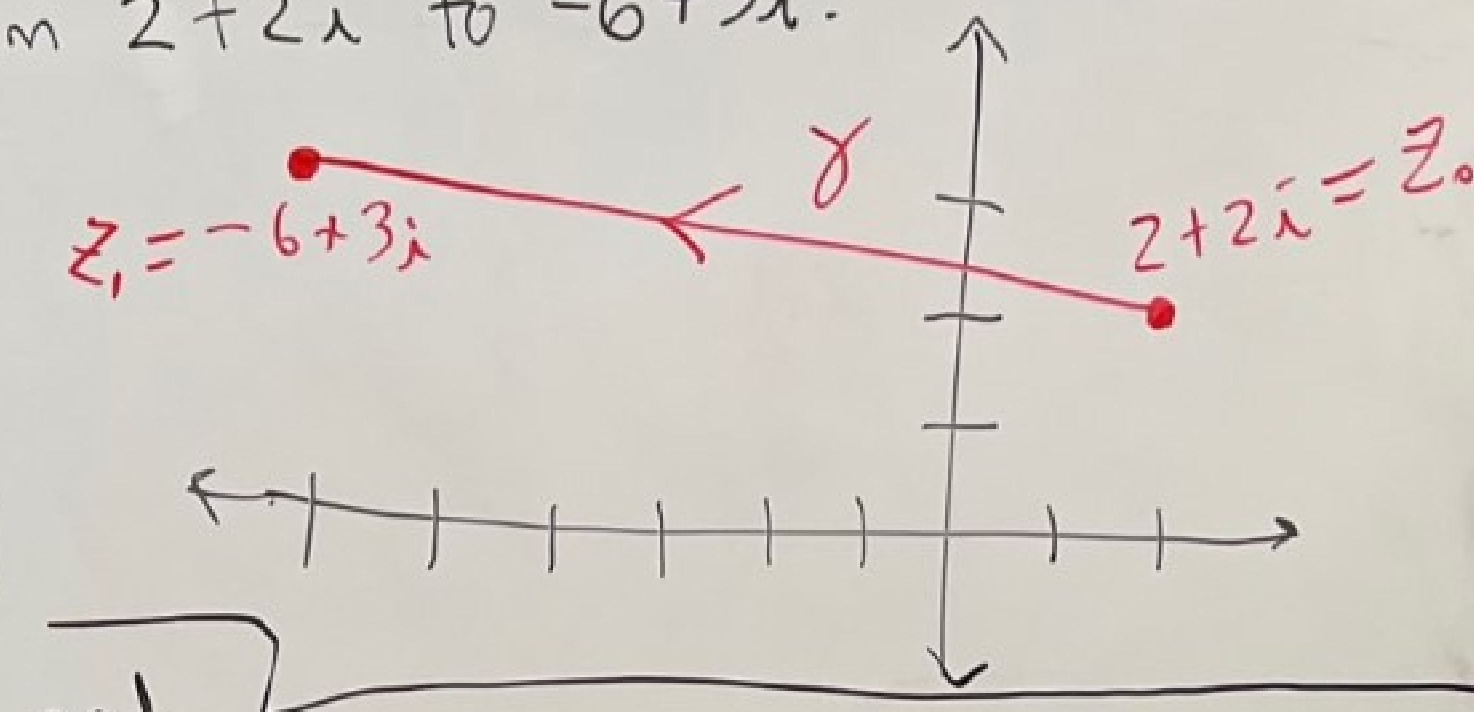
Parameterizing a straight line segment

The line segment from z_0 to z_1 can be parameterized as follows:

$$\boxed{\begin{gathered}\gamma(t) = z_0 + t(z_1 - z_0)\\ 0 \le t \le 1\end{gathered}}$$



Ex: Parameterize the line segment from $2+2i$ to $-6+3i$.



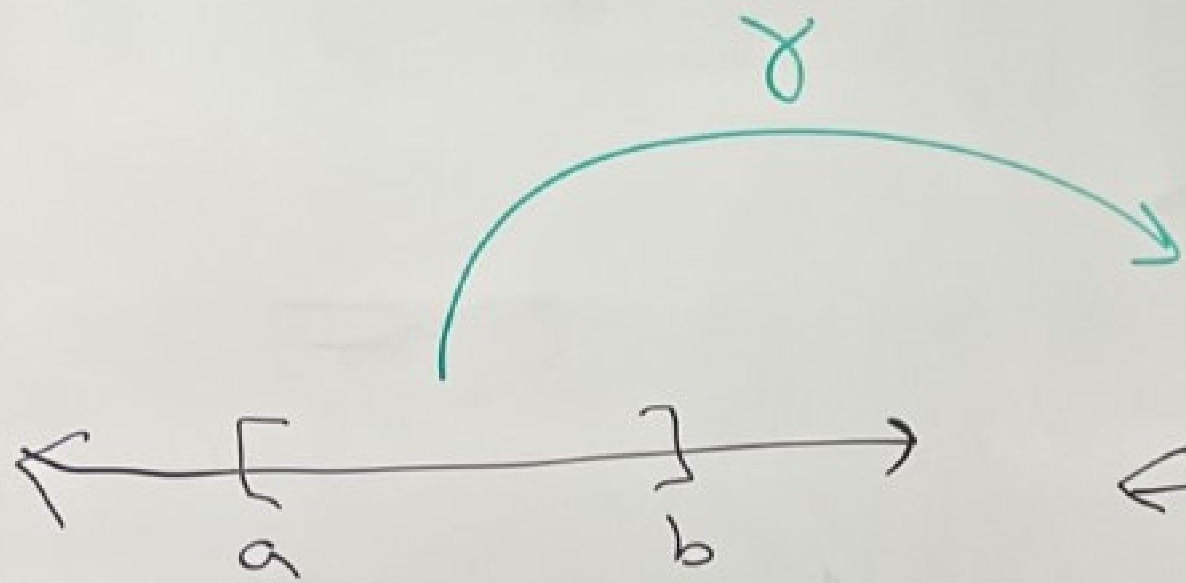
$$\gamma(t) = (2+2i) + t\left[(-6+3i) - (2+2i)\right]$$

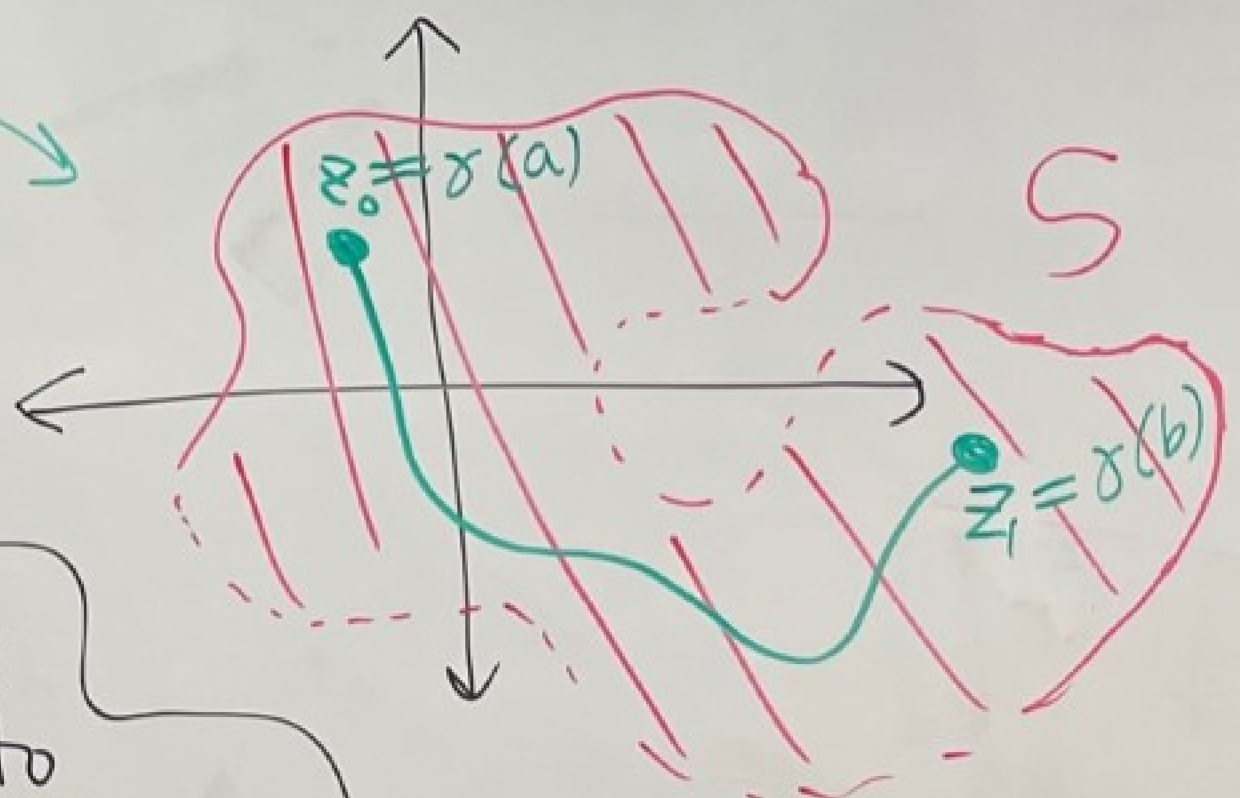
$$\gamma(t) = (2+2i) + t(-8+i), \quad 0 \le t \le 1$$

$$\gamma(t) = \underbrace{(2-8t)}_{u(t) = 2-8t} + i\underbrace{(2+t)}_{v(t) = 2+t}, \quad 0 \le t \le 1$$

$u'(t) = -8,\ v'(t) = 1 \longrightarrow \gamma$ is smooth and $\gamma'(t) = u'(t) + iv'(t) = -8 + i(1) = -8 + i$

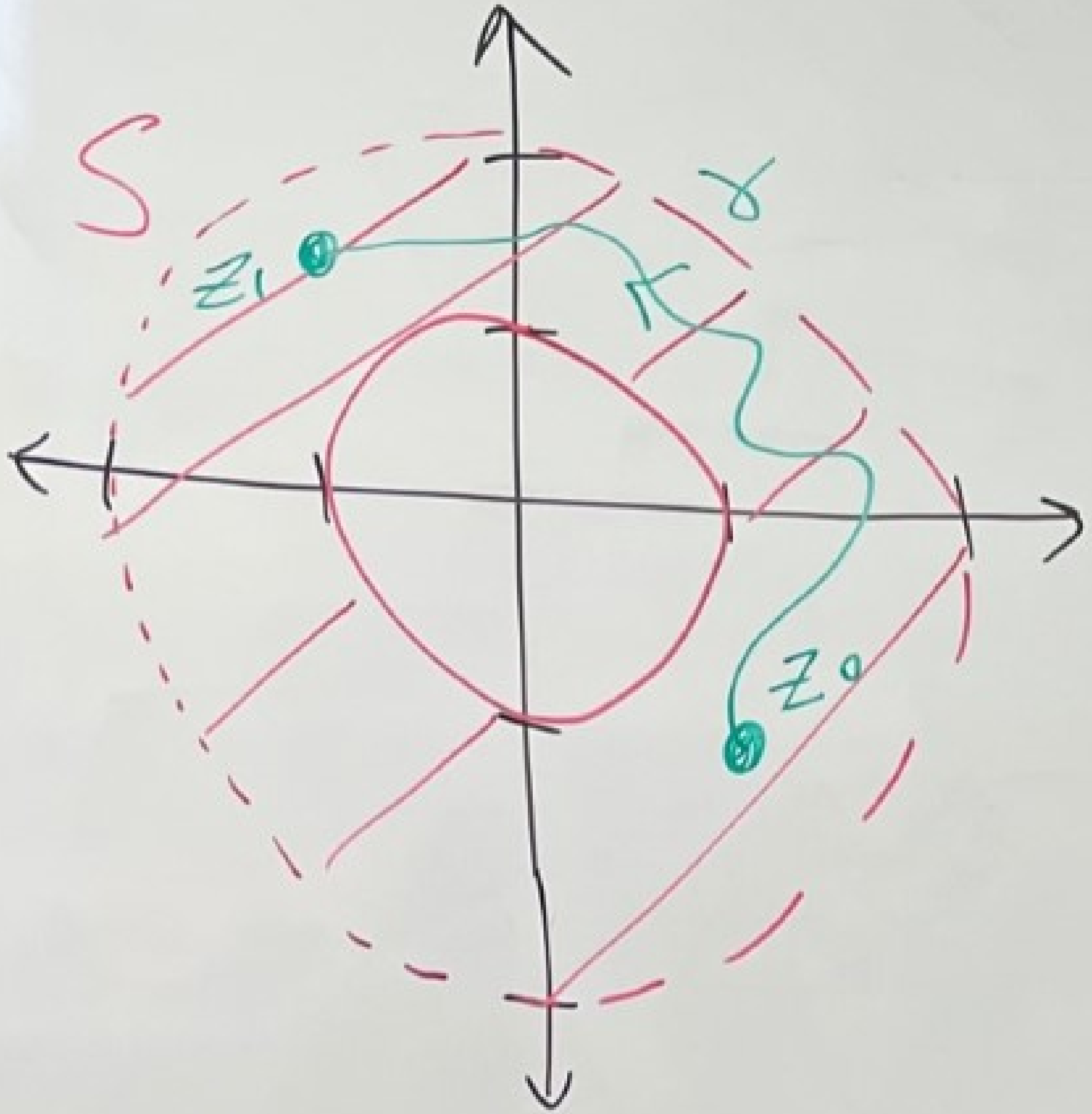
Def: Let $S \subseteq \mathbb{C}$. S is called **path-connected** if for every pair of points $z_0, z_1 \in S$ there exists a piecewise smooth curve $\gamma : [a,b] \longrightarrow S$ where $\gamma(a) = z_0$ and $\gamma(b) = z_1$.





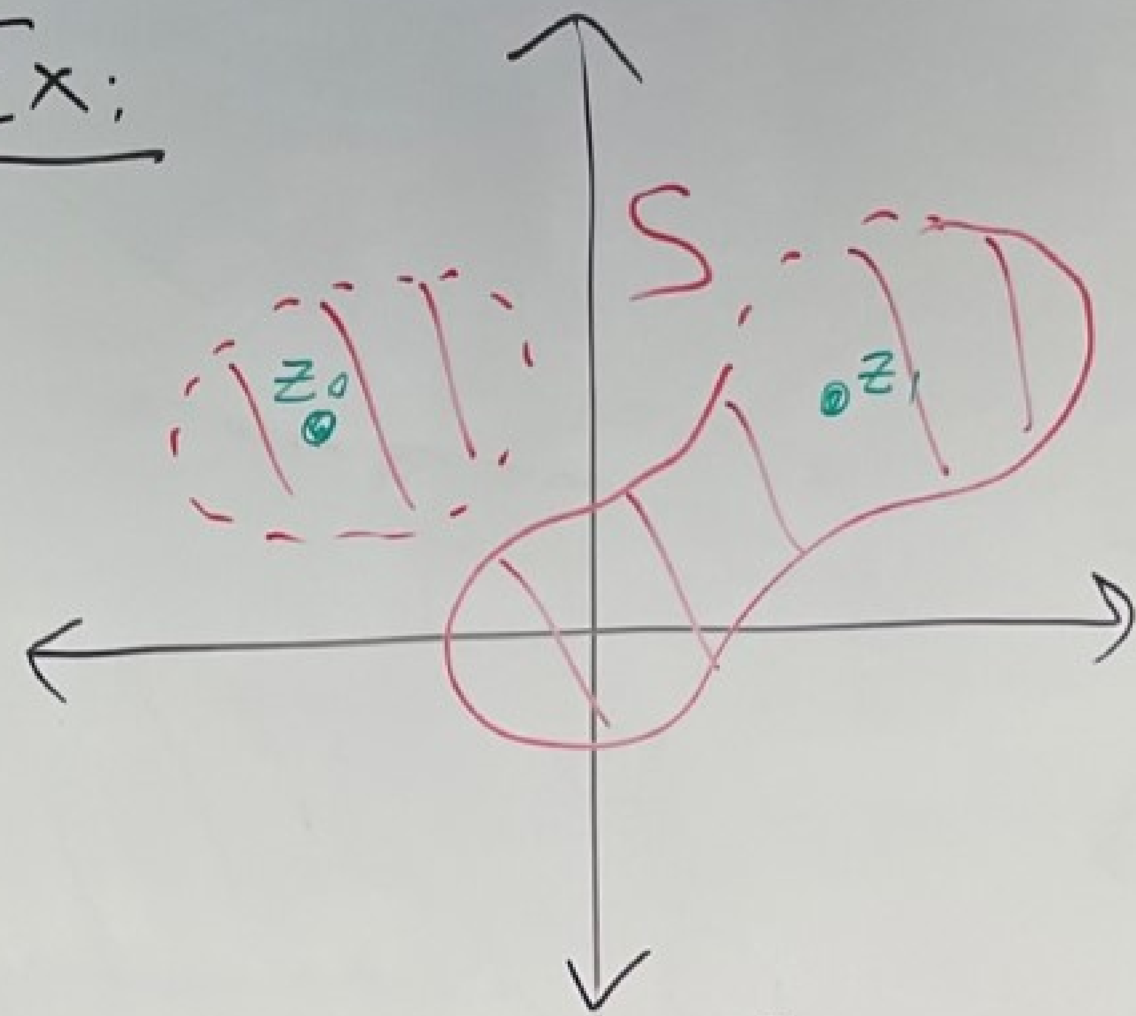
That is, any two points in S can be connected to each other by a piece-wise smooth curve that lies inside of S.

Ex: $S = \{ z \mid 1 \leq |z| < 2 \}$



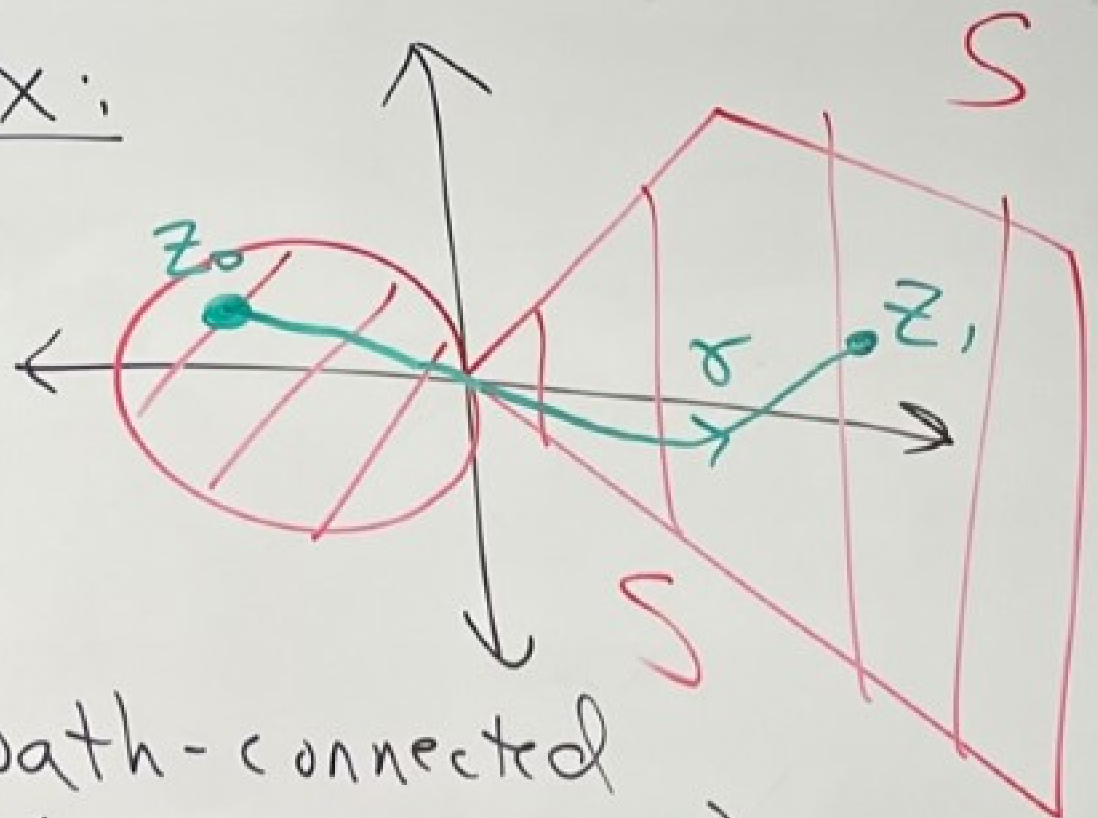
path-connected

Ex:



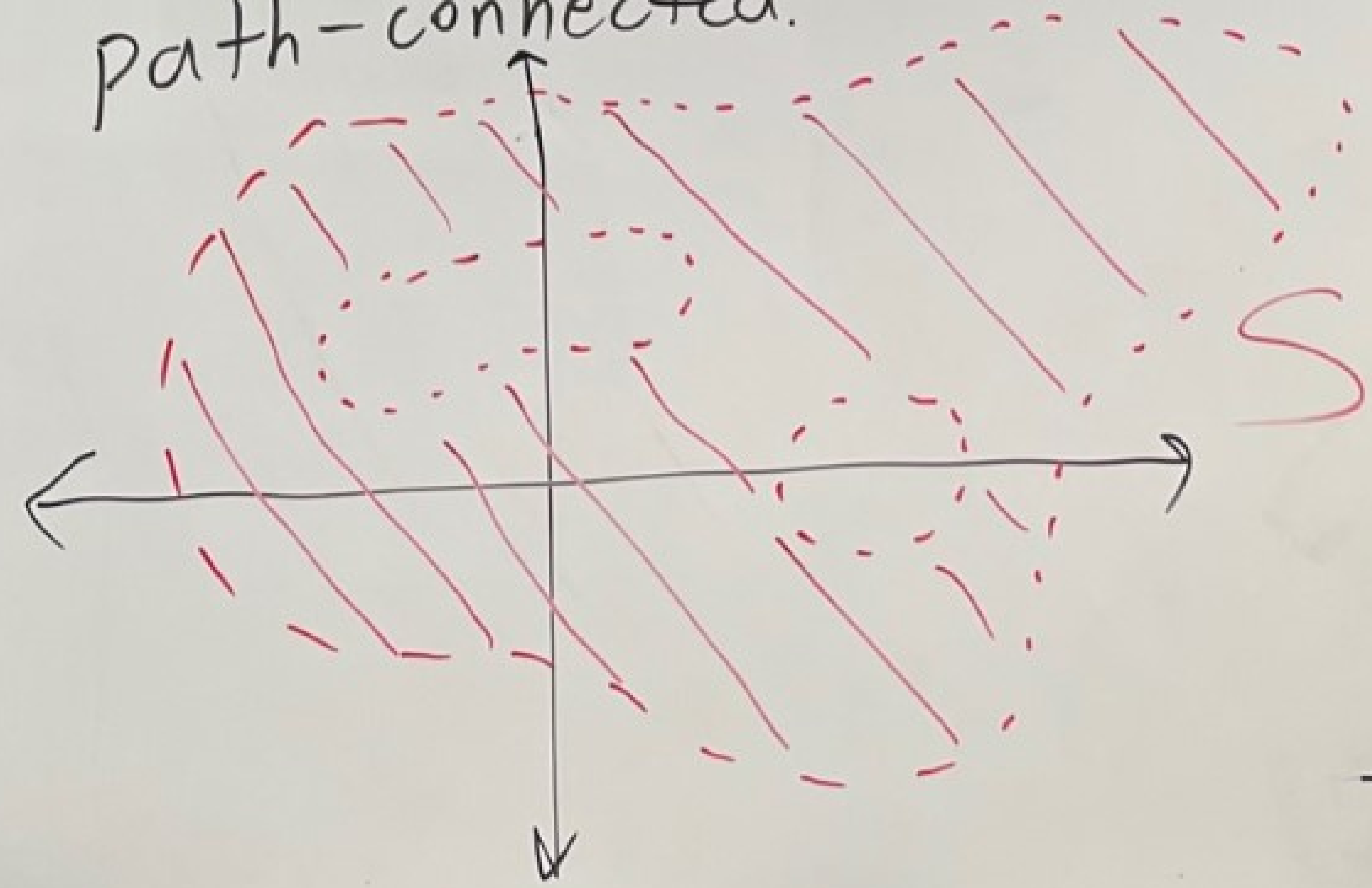
not path-connected
(can't get from z_0 to z_1 with a smooth curve that lies in S)

Ex:

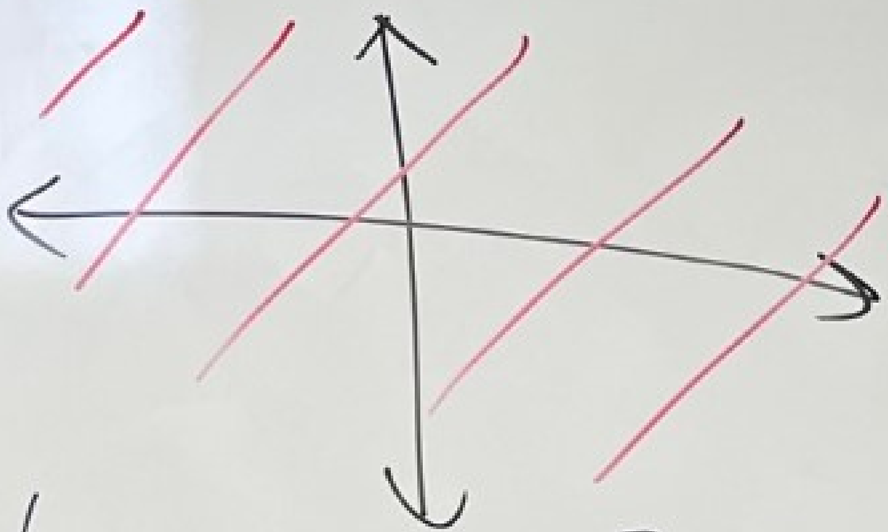


path-connected
(assuming $0 \in S$)

Def: Let $S \subseteq \mathbb{C}$.
We say that S is
a region (or domain)
if S is open and
path-connected.



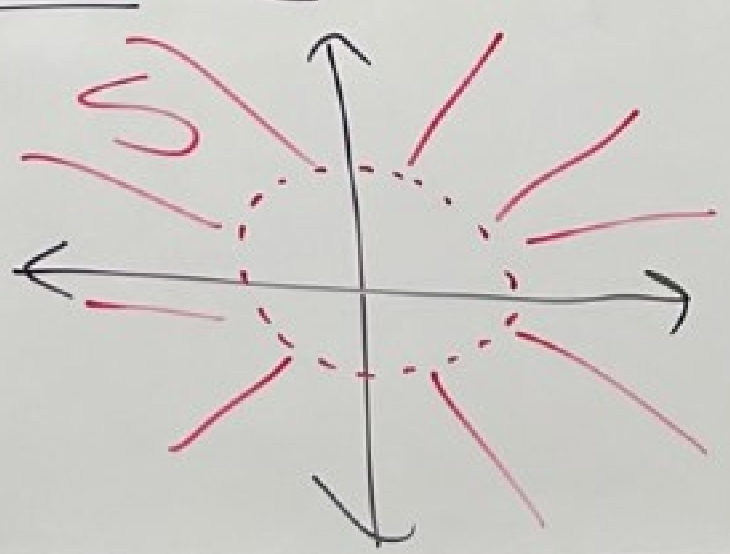
Ex: $S = \mathbb{C}$



✓ open
✓ path-connected

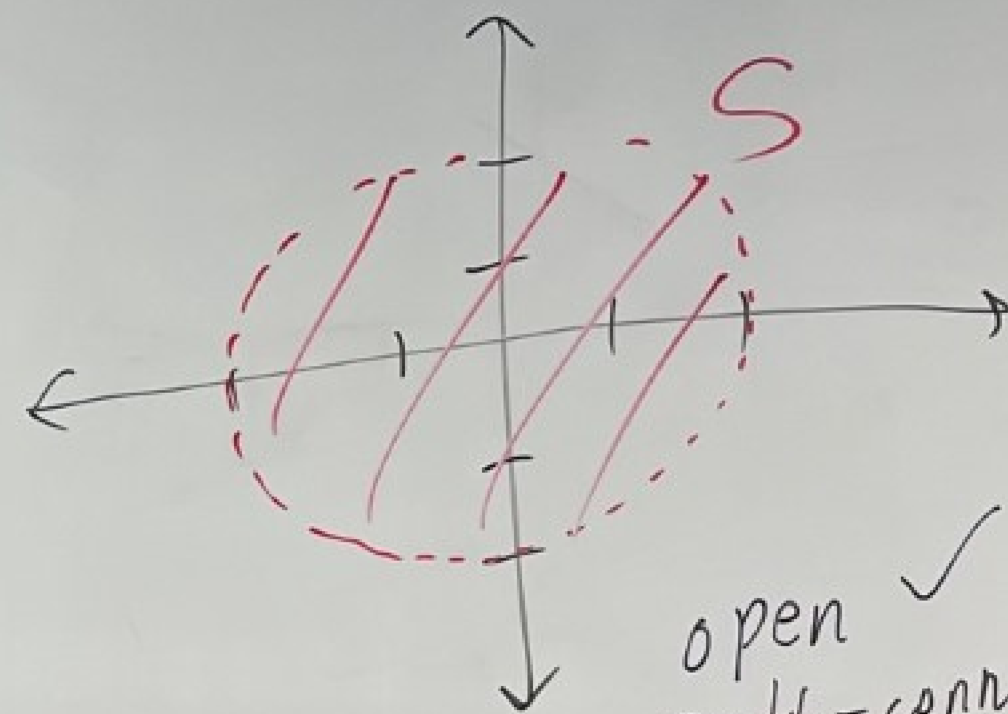
$\Big\}$ $S = \mathbb{C}$ is a region

Ex: $S = \mathbb{C} - \{ z \mid |z| \leq 1 \}$



open ✓
path-connected ✓
S is a region

Ex: $S = D(0; 2)$



open ✓
path-connected ✓

S is a region