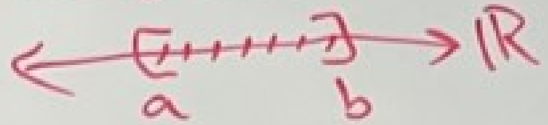
Def: Let $a, b \in \mathbb{R}$ and $a < b$.

Let $h : \underbrace{[a,b]} \longrightarrow \mathbb{C}$ where $h(t) = u(t) + i\,v(t)$



The __integral__ of h on $[a,b]$ is defined to be

$$\int_a^b h(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

calculus/real analysis integrals

Ex:

$$\int_0^2 \left[t^2 + i(t+1)\right] dt = \left(\int_0^2 t^2\,dt\right) + i\left(\int_0^2 (t+1)\,dt\right)$$

$$= \frac{t^3}{3}\Bigg|_0^2 + i\left[\frac{t^2}{2} + t\right]\Bigg|_0^2$$

$$= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) + i\left(\left(\frac{2^2}{2} + 2\right) - \left(\frac{0^2}{2} + 0\right)\right)$$

$$= \boxed{\frac{8}{3} + 4i}$$

Integral of $f(z)$

Let $f(z)$ be a function of a complex variable z.

Let γ be a piecewise smooth curve from A to B.

Suppose f is defined on all the points of γ.

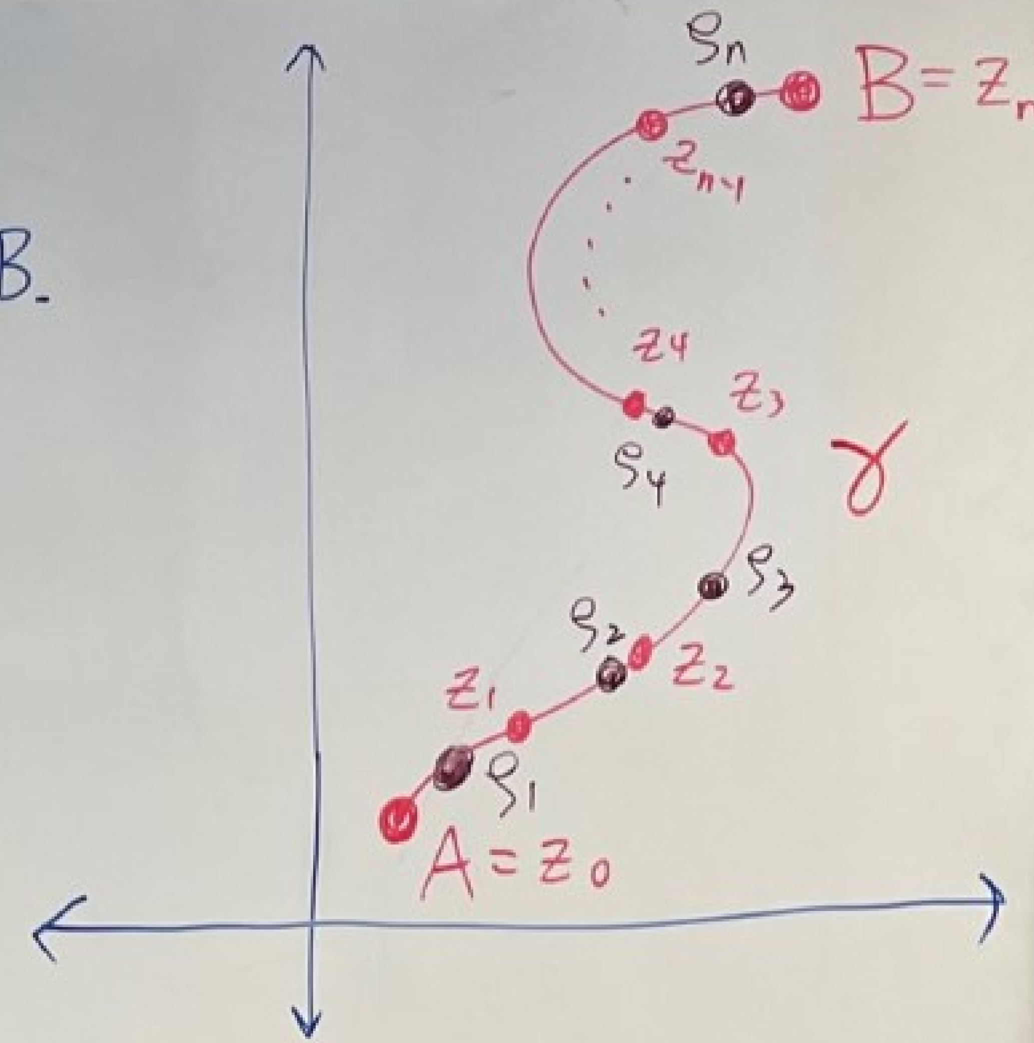
Let $z_0 = A$ and $z_n = B$ where $n \geq 2$.

Pick $n-1$ distinct points on γ between A and B.

Call them $z_1, z_2, \ldots, z_{n-1}$. Let $\Delta z_k = z_k - z_{k-1}$.

Let ξ_k be any point on γ between z_{k-1} and z_k.

So you can form the sum $\sum_{k=1}^{n} f(\xi_k)\Delta z_k$



Now let n increase without bound and for each n do a subdivision and

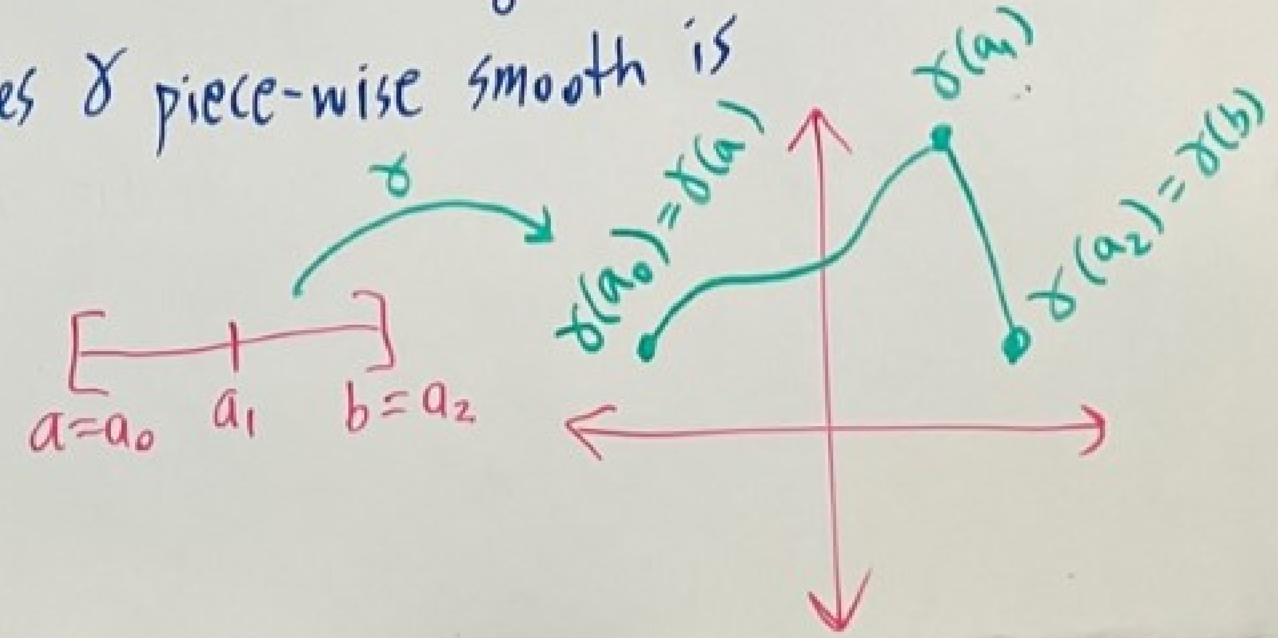
consider $\lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta z_k$ as $n \to \infty$ and $|\Delta z_k|$ approaches 0 for each k.

If this limit exists and is independent of the particular subdivisions chosen at each step then we define the limit to be the definite integral of f along γ and write

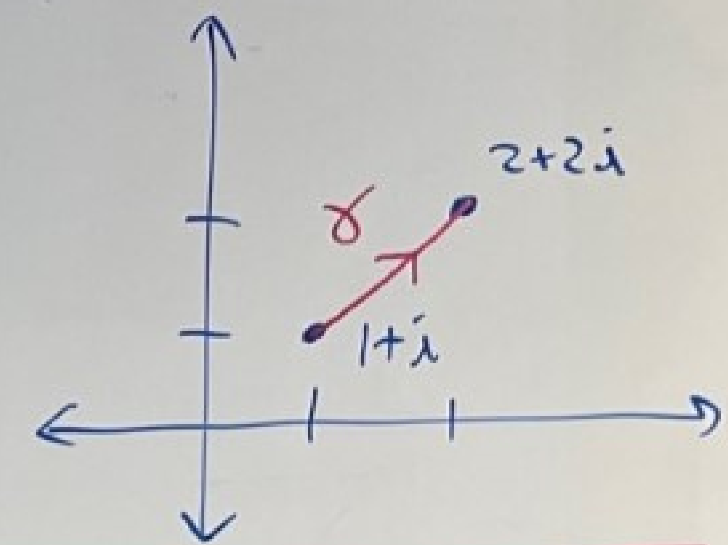
$$\int_\gamma f(z)\,dz = \lim_{\substack{n\to\infty \\ |\Delta z_k|\to 0}} \sum_{k=1}^{n} f(\xi_k)\,\Delta z_k$$

Theorem: Suppose that $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is open. Suppose f is continuous on A. Let $\gamma:[a,b] \to A$ be a piece-wise smooth curve. Then $\int_\gamma f(z)\,dz$ exists. Furthermore, if the partition of $[a,b]$ that makes γ piece-wise smooth is $a=a_0<a_1<\dots<a_n=b$ then

$$\int_\gamma f(z)\,dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\cdot\gamma'(t)\,dt$$



Ex: Integrate $f(z) = 2z+1$ on the line segment from $1+i$ to $2+2i$.



parameterize γ: $\gamma(t) = (1+i) + t[(2+2i)-(1+i)] = (1+i) + t(1+i)$

$\gamma(t) = (1+t) + i(1+t), \quad 0 \le t \le 1$

compute the integral: $\gamma'(t) = 1+i$

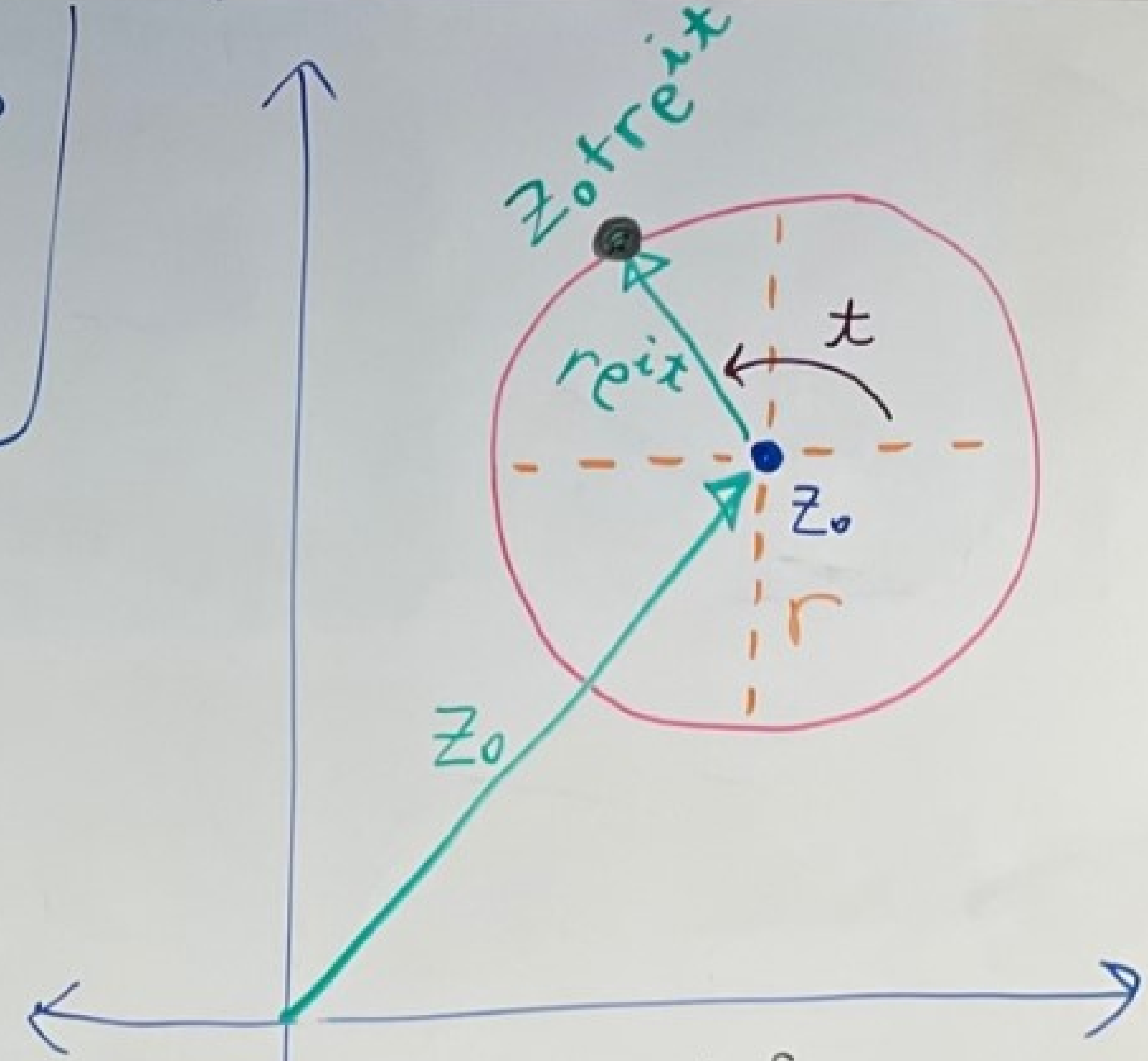
$$\int_\gamma f = \int_0^1 f(\gamma(t))\cdot\gamma'(t)\,dt = \int_0^1 \Big(\underbrace{2\big[(1+t)+i(1+t)\big]+1}_{f(\gamma(t))}\Big)\cdot\underbrace{(1+i)}_{\gamma'(t)}\,dt = \int_0^1 \underbrace{(2+2t+2i+2it+1)}_{3+2t+2i+2it}(1+i)\,dt$$

$$= \int_0^1 (3+2t+2i+2it+3i+2ti-2-2t)\,dt = \int_0^1 (1+5i+4ti)\,dt$$

$$= \int_0^1 1\,dt + i\int_0^1 (5+4t)\,dt = t\Big|_0^1 + i\left(5t+\frac{4t^2}{2}\right)\Big|_0^1 = (1-0)+i\big((5+2)-0\big) = \boxed{1+7i}$$

Parameterization for a circle centered at z_0 with radius r, going around once in the counter-clockwise direction.

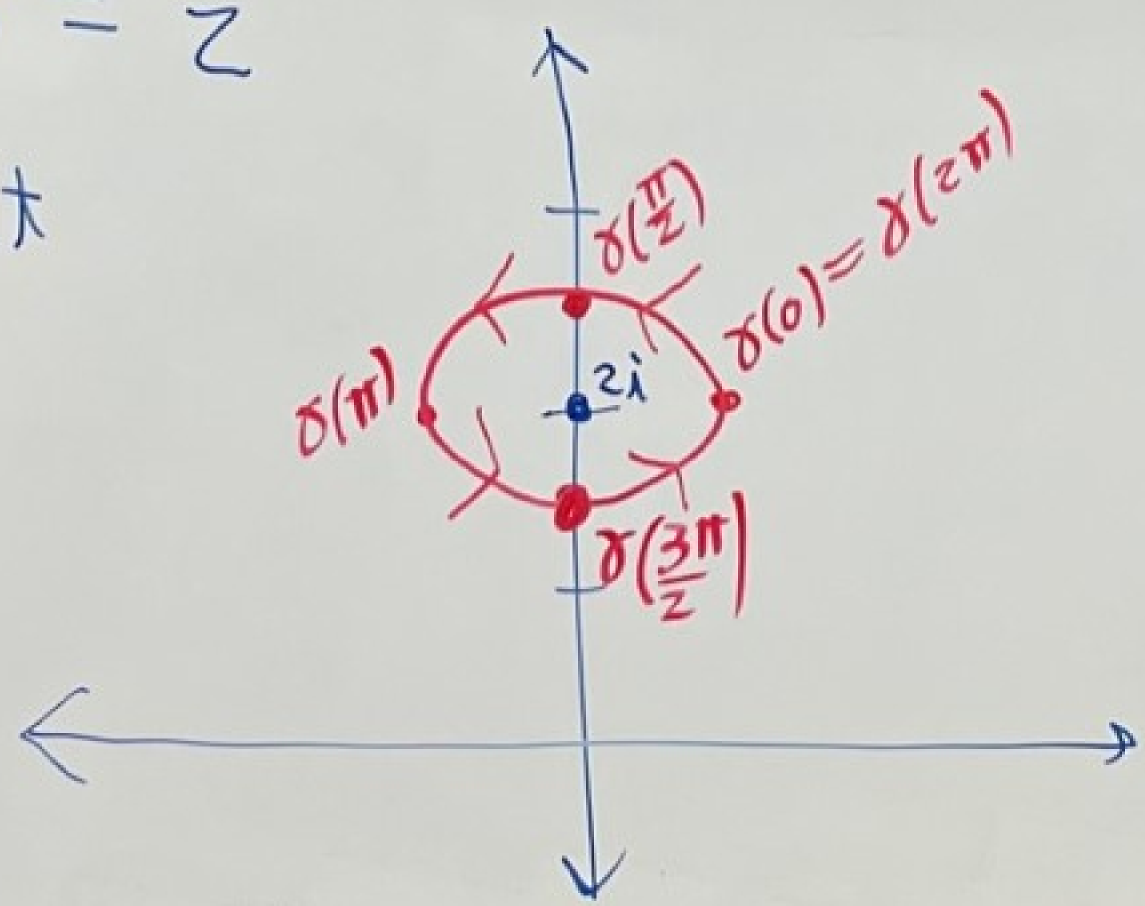
$$\boxed{\gamma(t) = z_0 + re^{it} \quad 0 \leq t \leq 2\pi}$$



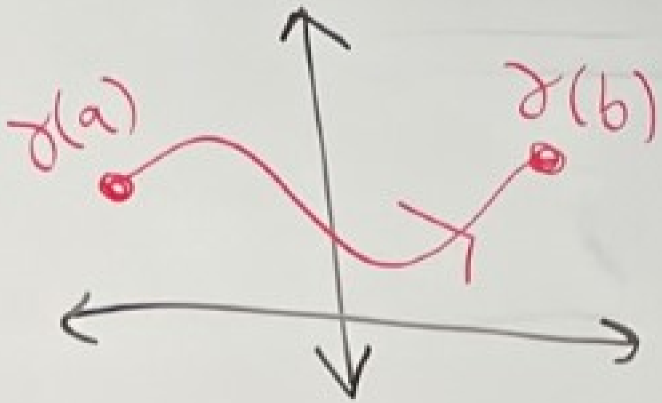
You are shifting the circle of radius r centered at 0 over to z_0.

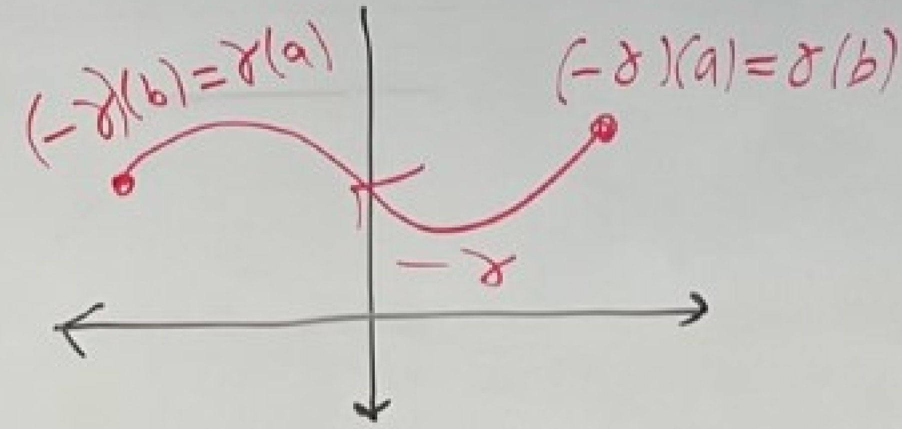
Ex: $z_0 = 2i$, $r = \frac{1}{2}$

$$\gamma(t) = 2i + \frac{1}{2}e^{it}$$
$$0 \leq t \leq 2\pi$$



Def: For a curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ we define the opposite curve $-\gamma : [a,b] \longrightarrow \mathbb{C}$ by setting $(-\gamma)(t) = \gamma(a+b-t)$





$\boxed{-\gamma \text{ reverses the direction of } \gamma}$

$$\boxed{\begin{aligned} (-\gamma)(a) &= \gamma(a+b-a) = \gamma(b) \\ (-\gamma)(b) &= \gamma(a+b-b) = \gamma(a) \end{aligned}}$$