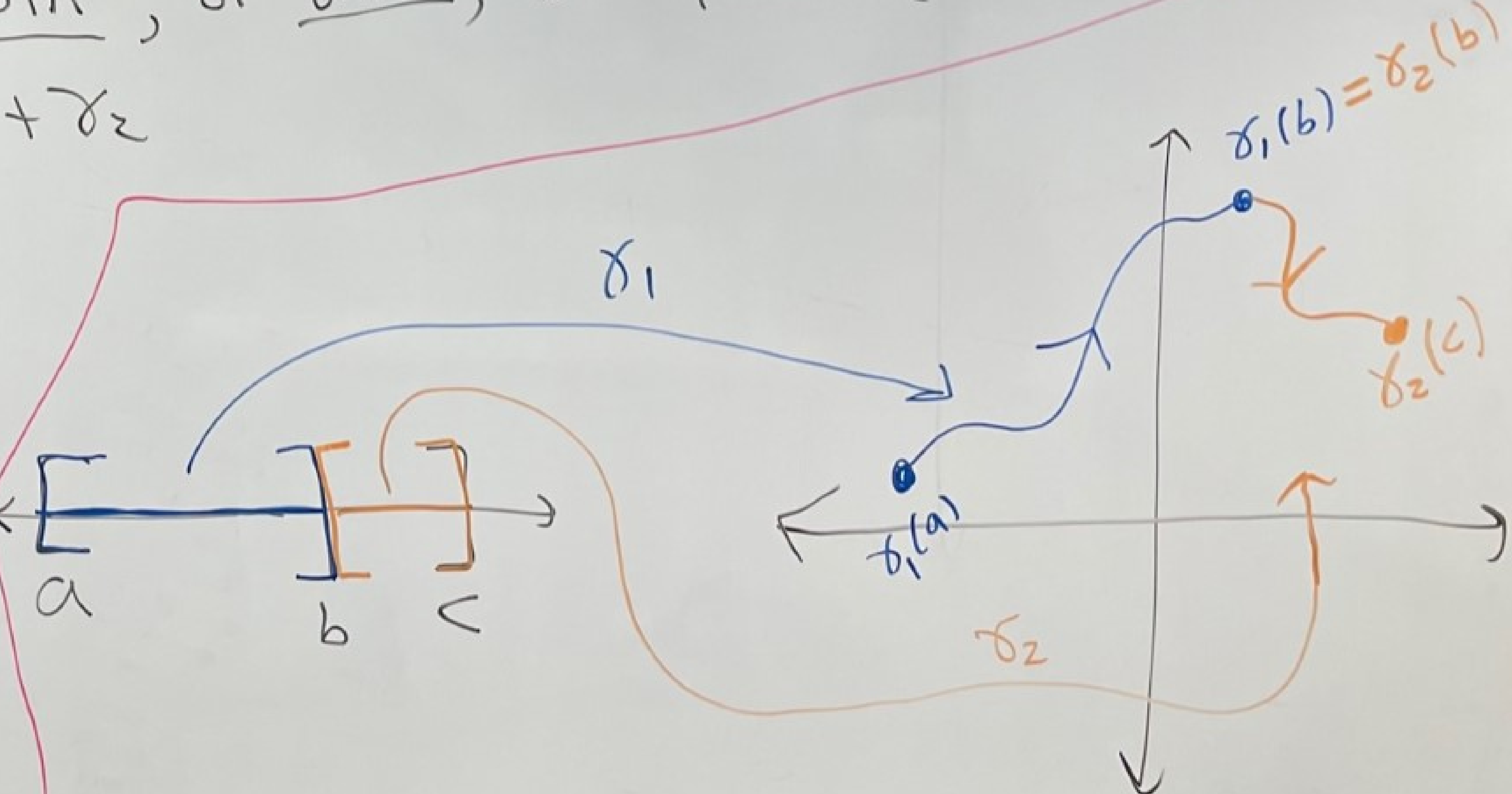
Def: Suppose $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ are two curves with $\gamma_1(b) = \gamma_2(b)$.

The sum, or join, or union, of γ_1 and γ_2 is denoted by $\gamma_1 + \gamma_2$ is defined as

$$(\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t), & a \le t \le b \\ \gamma_2(t), & b \le t \le c \end{cases}$$

Here $\gamma_1 + \gamma_2 : [a,c] \to \mathbb{C}$



Theorem: Let $\alpha, \beta \in \mathbb{C}$. Let f and g be functions that are continuous on some open set containing some piece-wise smooth curves $\gamma_1, \gamma_2, \gamma$ where γ_1 ends where γ_2 begins.

Then:

(1) $$\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$$

(2) $$\int_{-\gamma} f = -\int_\gamma f$$

(3) $$\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

proof:

You can try or look at Hoffman/Marsden book. ∎

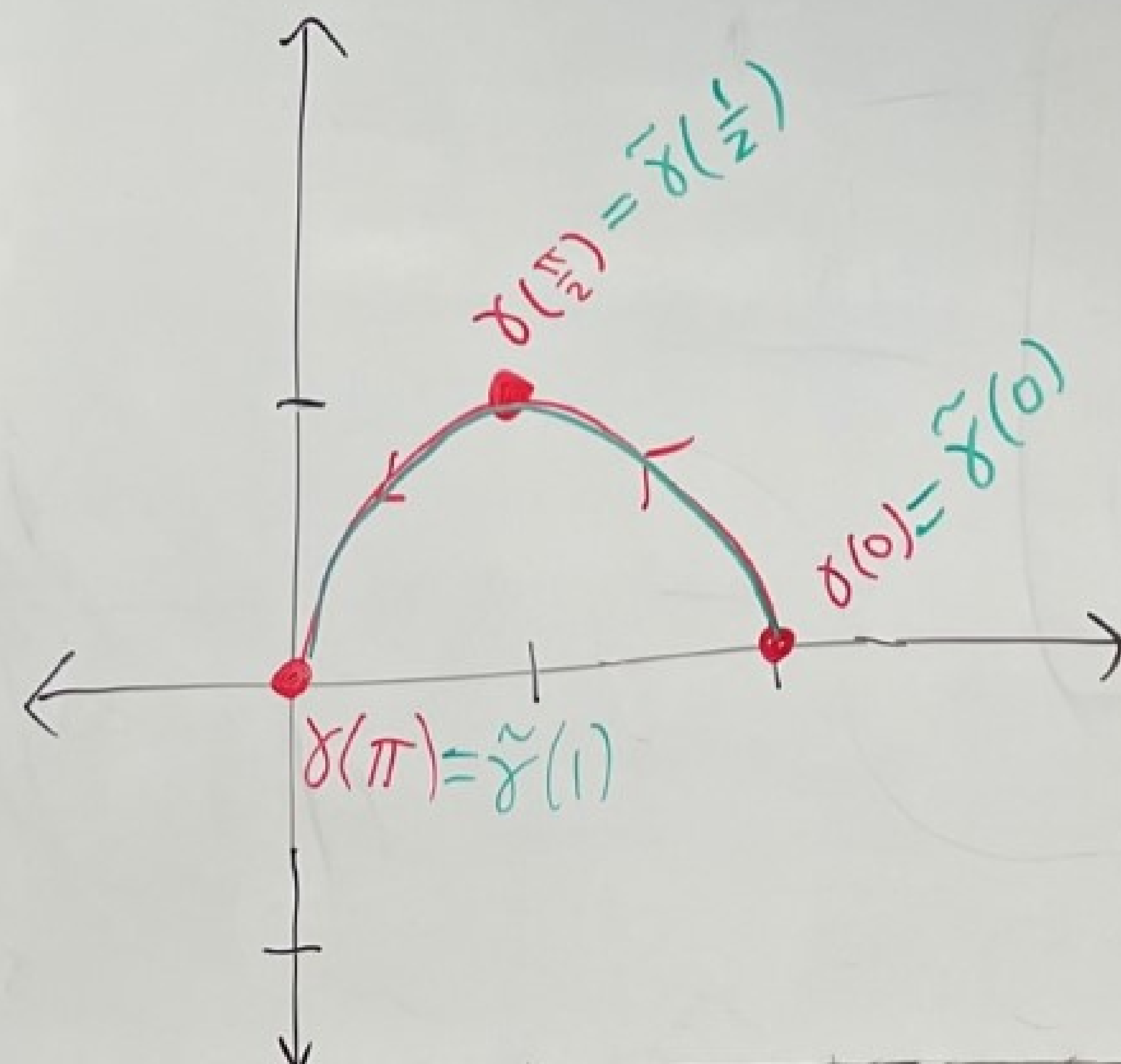
Theorem: If γ and $\tilde{\gamma}$ are "parameterizations" of the same curve, then $\int_{\gamma} f = \int_{\tilde{\gamma}} f$.

$\boxed{\text{See Hoffman/Marsden book for more info}}$

Ex:

$$\gamma(t) = 1 + e^{it}, \quad 0 \le t \le \pi$$

$$\tilde{\gamma}(t) = 1 + e^{i\pi t}, \quad 0 \le t \le 1$$



γ and $\tilde{\gamma}$ parameterize the same curve.

So, $\int_{\gamma} f = \int_{\tilde{\gamma}} f$

Def: (arclength of a curve)

Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a smooth curve where $\gamma(t) = u(t) + iv(t)$.

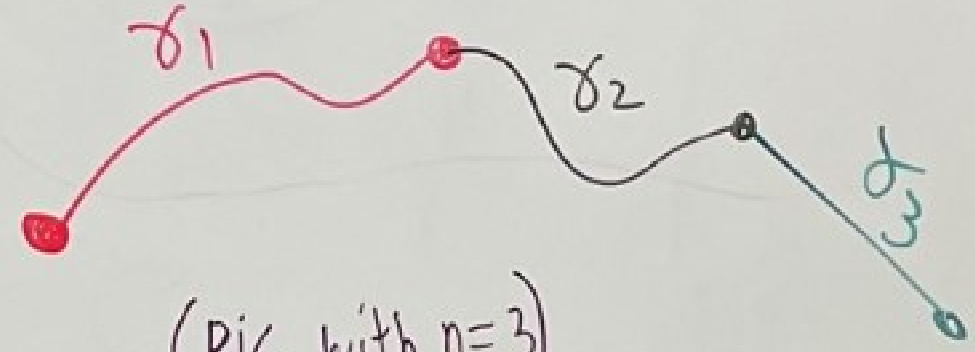
The arclength of γ is defined to be

$$\text{arclength}(\gamma) = \int_a^b |\gamma'(t)|\,dt = \int_a^b \sqrt{(u'(t))^2 + (v'(t))^2}\,dt$$

If γ is piecewise smooth, that is $\gamma = \sum_{i=1}^{n} \gamma_i$ where γ_i are each smooth curves (where the endpoint of γ_i equals the starting point of γ_{i+1})

Then define

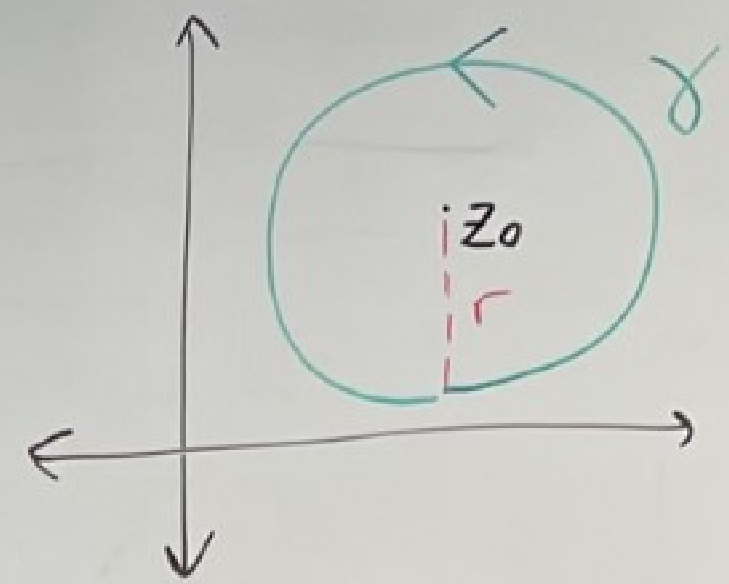
$$\text{arclength}(\gamma) = \sum_{i=1}^{n} \text{arclength}(\gamma_i)$$



(pic with $n=3$)

Ex: Consider the smooth curve $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

Let $z_0 = x_0 + iy_0$



So, $\gamma(t) = x_0 + iy_0 + r[\cos(t) + i\sin(t)]$

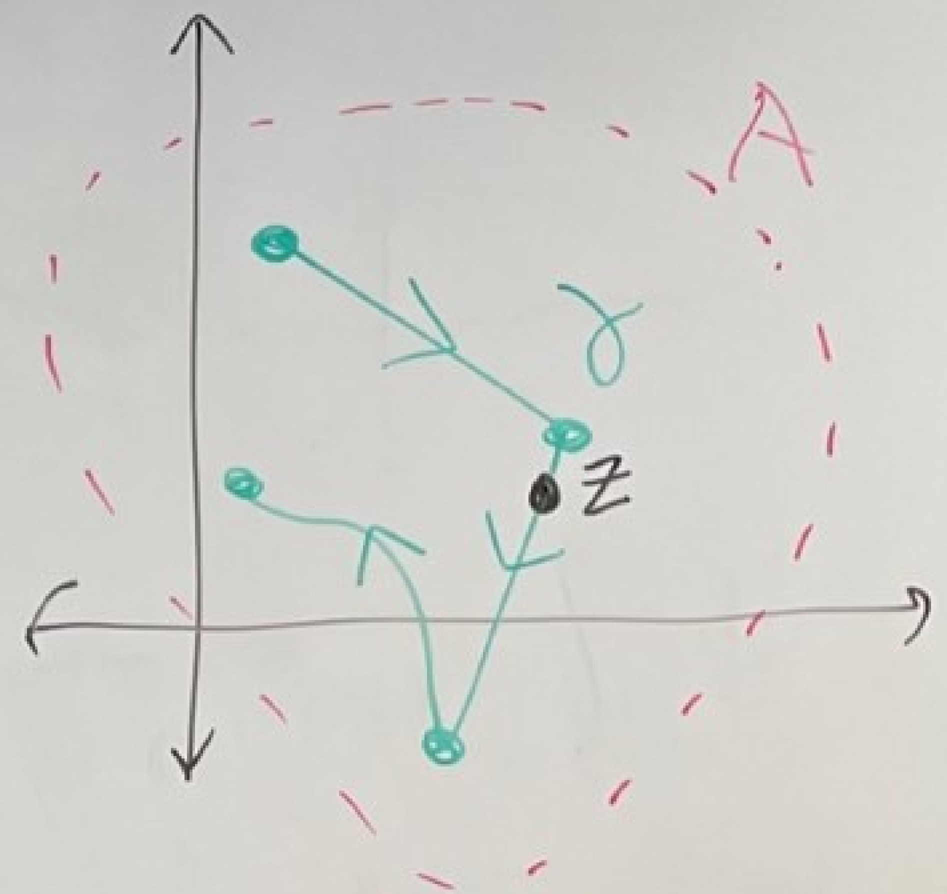
$$= \underbrace{(x_0 + r\cos(t))}_{u(t) = x_0 + r\cos(t)} + i\underbrace{(y_0 + r\sin(t))}_{v(t) = y_0 + r\sin(t)}$$

Then, $\gamma'(t) = \underbrace{-r\sin(t)}_{u'(t)} + i\,\underbrace{r\cos(t)}_{v'(t)}$.

$$\text{arclength}(\gamma) = \int_0^{2\pi} \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}\,dt = \int_0^{2\pi} r\sqrt{\sin^2(t) + \cos^2(t)}\,dt = \int_0^{2\pi} r\,dt$$

$$= rt\Big|_0^{2\pi} = r[2\pi - 0] = \boxed{2\pi r}$$

Theorem: Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is an open set. Suppose f is continuous on A. Let γ be a piecewise smooth curve whose image is contained in A.



Suppose there exists a real-number $M > 0$, where $|f(z)| \leq M$ for all z on γ.

Then,
$$\left| \int_\gamma f \right| \leq M \cdot \text{arclength}(\gamma)$$

Proof: For simplicity, let's assume γ is a smooth curve.

Then, $\displaystyle\int_\gamma f = \int_a^b \underbrace{f(\gamma(t))\cdot\gamma'(t)}_{g(t)}\,dt$ where $\gamma:[a,b]\to\mathbb{C}$, $a<b$.

Let $g(t) = f(\gamma(t))\cdot\gamma'(t)$.

We have $\displaystyle\int_\gamma f = \int_a^b g(t)\,dt = re^{i\theta}$ for some $r,\theta\in\mathbb{R}$, $r>0$.

Then, $\displaystyle r = e^{-i\theta}\int_a^b g(t)\,dt = \int_a^b e^{-i\theta} g(t)\,dt$

So,

$$r \underset{r\in\mathbb{R}}{=} \mathrm{Re}(r) = \mathrm{Re}\left(\int_a^b e^{-i\theta} g(t)\,dt\right) = \int_a^b \mathrm{Re}\left(e^{-i\theta} g(t)\,dt\right)$$

$$\boxed{\begin{aligned} &\mathrm{Re}\left(\int_a^b (u(t)+iv(t))\,dt\right)\\ &= \mathrm{Re}\left(\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\right)\\ &= \int_a^b u(t)\,dt = \int_a^b \mathrm{Re}\left(u(t)+iv(t)\right)dt \end{aligned}}$$

From class, we know $\boxed{\mathrm{Re}(z) \leq} |\mathrm{Re}(z)| \leq \boxed{|z|}$ for all $z \in \mathbb{C}$

Thus, $\mathrm{Re}(e^{-i\theta} g(t)) \leq |e^{-i\theta} g(t)| = \underbrace{|e^{-i\theta}|}_{1} |g(t)| = |g(t)|$

So, $r = \int_a^b \underbrace{\mathrm{Re}(e^{-i\theta} g(t))}_{\text{this is a real number}} dt \underset{\substack{\uparrow \\ \text{4650 calculus bound}}}{\leq} \int_a^b \underbrace{|g(t)|}_{\text{real number}} dt$

Therefore,

$$\left|\int_\gamma f\right| = \left|\int g(t)\,dt\right| = |re^{i\theta}| = r \leq \int_a^b |g(t)|\,dt = \int_a^b |f(\gamma(t)) \cdot \gamma'(t)|\,dt$$

$$= \int_a^b \underbrace{|f(\gamma(t))|}_{\leq M} \cdot |\gamma'(t)|\,dt \underset{\substack{\uparrow \\ \text{4650 calculus bound}}}{\leq} \int_a^b M \cdot |\gamma'(t)|\,dt = M \cdot \int_a^b |\gamma'(t)|\,dt$$

$$= M \cdot \mathrm{arclength}(\gamma).$$

$\blacksquare$