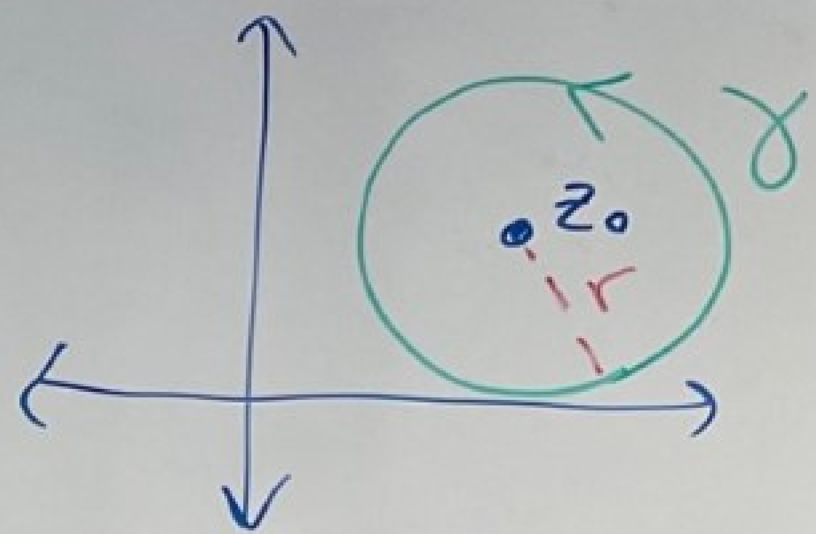
Ex (continued...) $[r>0,\ r\in\mathbb{R}]$



$$\int_\gamma (z-z_0)^n\,dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

proof: We did case 1 $(n \geq 0)$ last time.

case 2: Suppose $n \leq -2$.

Do the same idea as case 1.

Let $F(z) = \frac{1}{n+1}(z-z_0)^{n+1}$. Then $n+1 \leq -1$. So, $F(z) = \frac{1}{n+1}\,\frac{1}{(z-z_0)^{-n-1}}$ where $-n-1 \geq 1$.

Then, F is analytic on $\mathbb{C} - \{z_0\}$ $\leftarrow$ (This means $F'(z)$ exists except at $z = z_0$)

So, F is analytic on γ and $F'(z) = (z-z_0)^n$ which is continuous on γ.

So, by FTOC $\int (z-z_0)^n\,dz = F(\text{end point of } \gamma) - F(\text{start point of } \gamma) \overset{\uparrow}{=} 0$

$\boxed{\gamma \text{ is a closed curve}}$

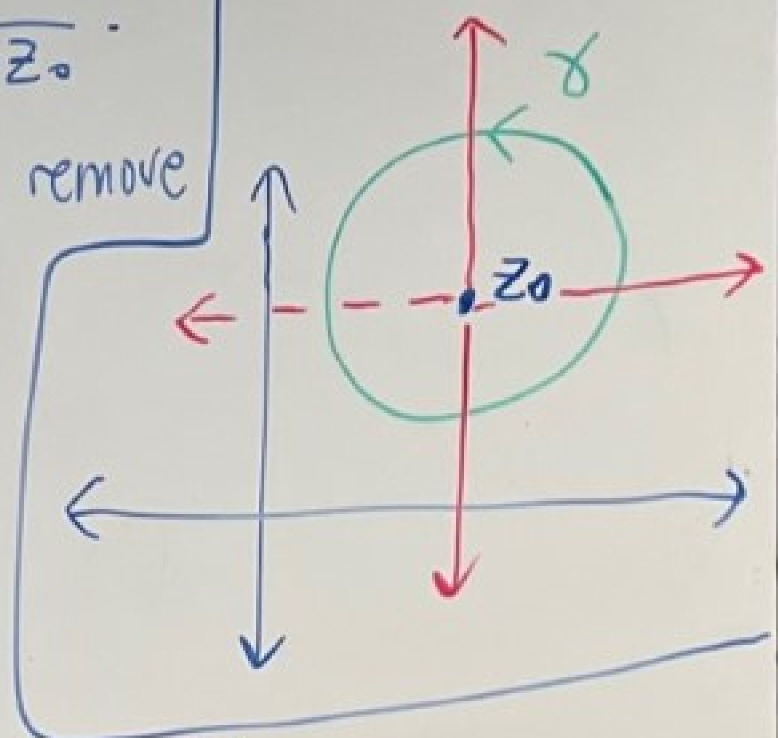
Case 3: Suppose $n=-1$.

So we want to calculate $\displaystyle\int_\gamma \frac{1}{z-z_0}\,dz$

You might be tempted to try $F(z)=\log(z-z_0)$ since $F'(z)=\dfrac{1}{z-z_0}$.

But to calculate F' you need a branch of log and the part you remove will hit the curve γ. So you can't use FTOC.

$\log(z-z_0)$ would be differentiable except on the line you remove.



Let's calculate the integral directly.

Need to parameterize γ.

Let $\boxed{\gamma(t)=z_0+re^{it},\ 0\le t\le 2\pi}$

Let $z_0=x_0+iy_0$.

Then, $\gamma(t)=x_0+iy_0+r\cos(t)+ir\sin(t)=\left[x_0+r\cos(t)\right]+i\left[y_0+r\sin(t)\right]$

Thus, $\gamma'(t)=-r\sin(t)+ir\cos(t)=i\left[r\cos(t)+ir\sin(t)\right]=ire^{it}$

Ergo,

$$\int_\gamma \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \underbrace{\frac{1}{(z_0+re^{it})-z_0}}_{\text{plug } \gamma \text{ into } 1/(z-z_0)} \cdot \underbrace{ire^{it}}_{\gamma'(t)}\,dt = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\,dt$$

$$= \int_0^{2\pi} i\,dt = i\int_0^{2\pi} 1\,dt$$

$$= it\Big|_0^{2\pi} = i(2\pi-0) = 2\pi i$$

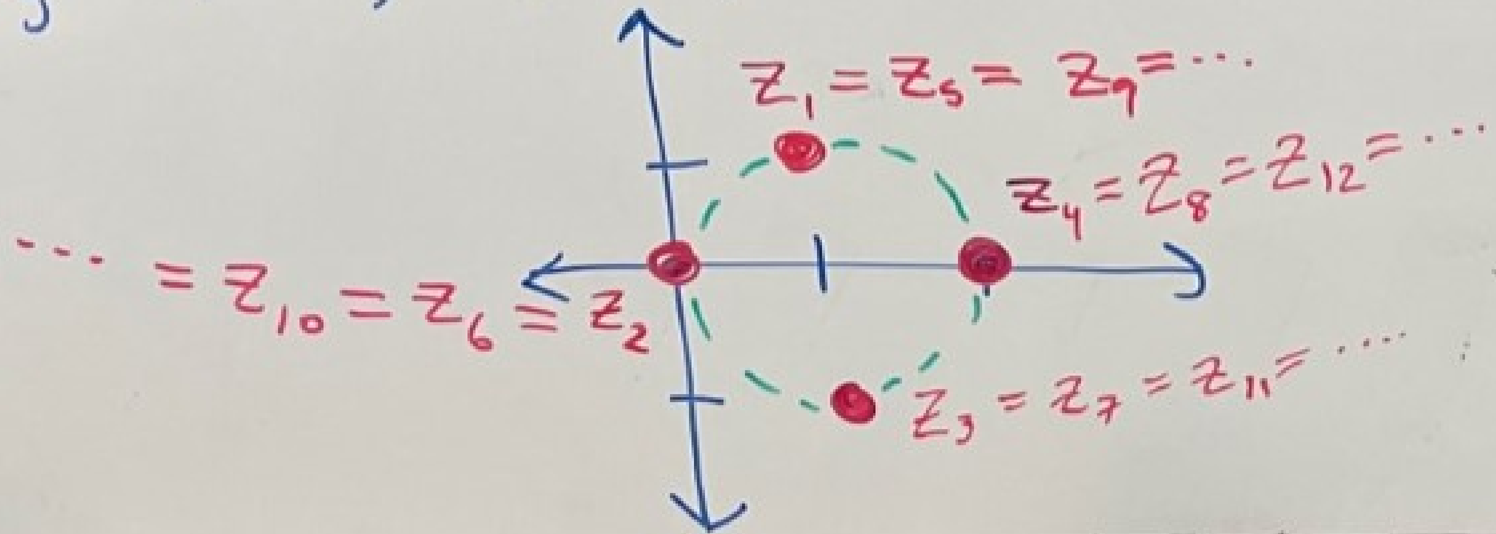
Topic 8 - Sequences

Def: A <u>sequence</u> $(z_n)_{n=1}^{\infty}$ is an ordered infinite list of complex numbers.

Ex: $z_n = 1 + i^n$

$z_1 = 1+i,\ z_2 = 0,\ z_3 = 1-i,\ z_4 = 2$

$z_5 = 1+i,\ z_6 = 0,\ z_7 = 1-i,\ z_8 = 2, \dots$



$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

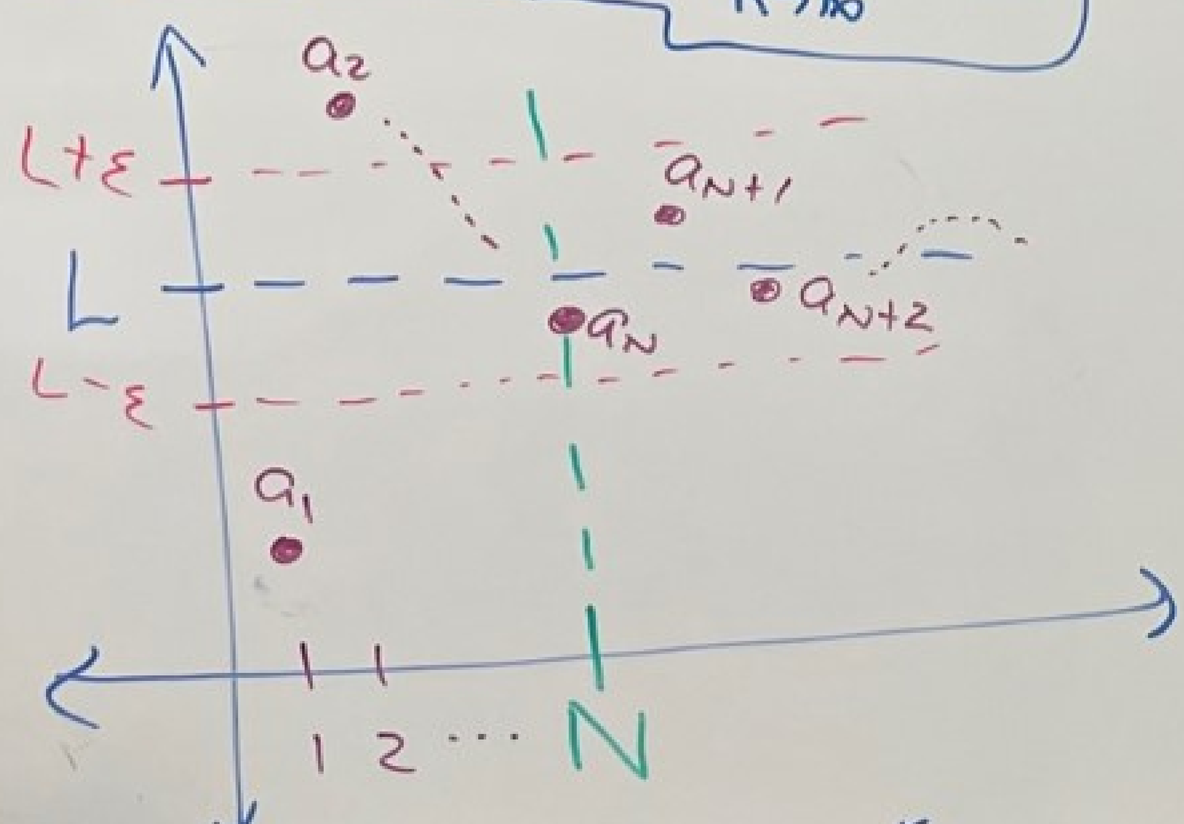
$i^7 = -i$

$i^8 = 1$

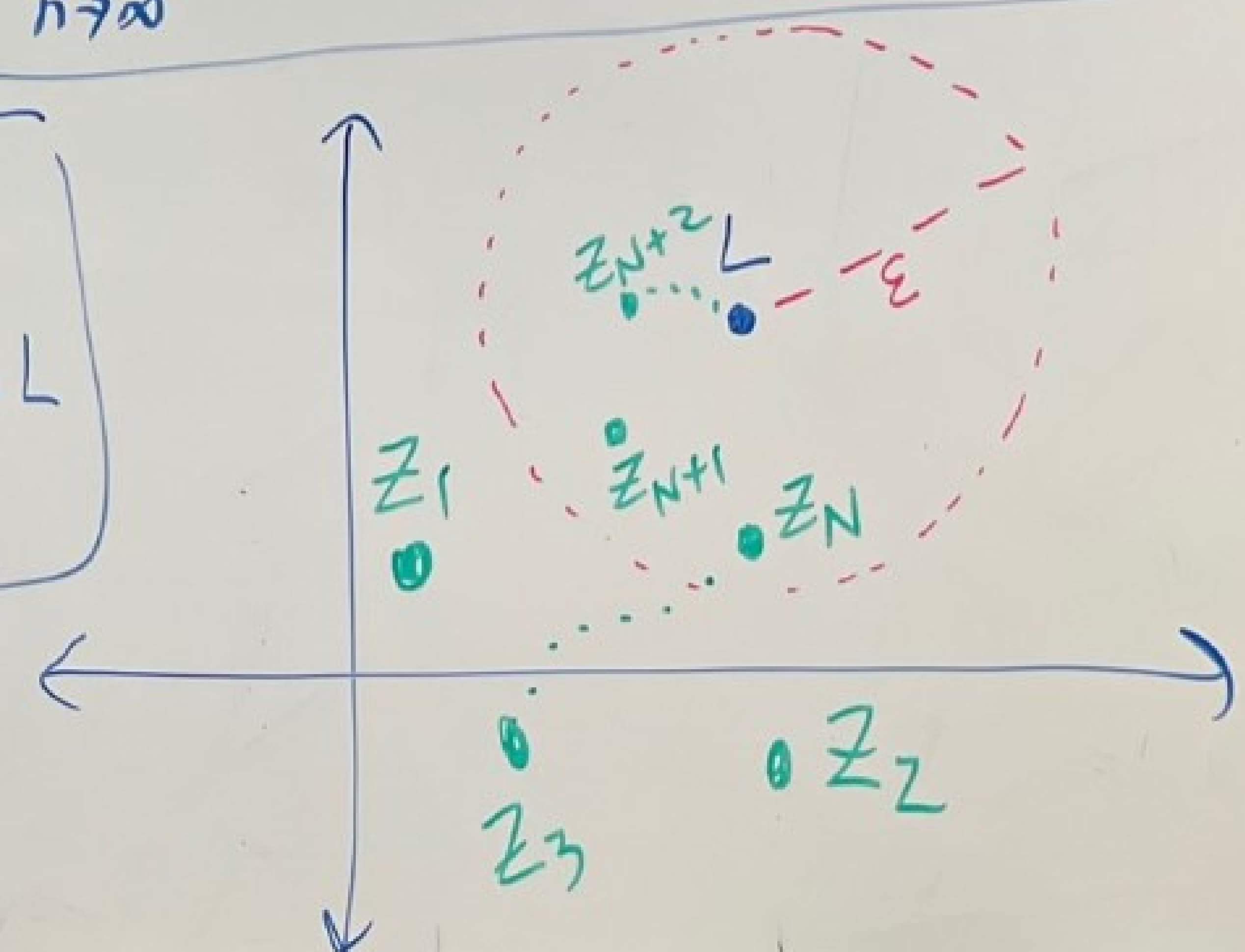
$\vdots \quad \vdots$

Def: Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. We say that $(z_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{C}$ if for every $\varepsilon > 0$ there exists $N > 0$ where if $n \geq N$ then $|z_n - L| < \varepsilon$. If this is the case then we write $\lim_{n\to\infty} z_n = L$ or $z_n \to L$ as $n \to \infty$

4650 picture for $\lim_{n\to\infty} a_n = L$



4680 picture
$\lim_{n\to\infty} z_n = L$



<u>Theorem:</u> Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers.
Let $L \in \mathbb{C}$.
Suppose $z_n = x_n + i y_n$ and $L = X + iY$.
Then,

$$\boxed{\lim_{n\to\infty} z_n = L} \quad \text{iff} \quad \boxed{\lim_{n\to\infty} x_n = X \text{ and } \lim_{n\to\infty} y_n = Y}$$

4680 limit

4650 limits

proof:

$(\Leftarrow)$ Suppose $\lim_{n\to\infty} x_n = X$ and $\lim_{n\to\infty} y_n = Y$.

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} x_n = X$ there exists $N_1 > 0$ where $n \geq N_1$ then $|x_n - X| < \varepsilon/2$

Since $\lim_{n\to\infty} y_n = Y$ there exists $N_2 > 0$ where if $n \geq N_2$ then $|y_n - y| < \varepsilon/2$

Let $N = \max\{N_1, N_2\}$.

Thus, if $n \geq N$, then $|z_n - L| = |x_n + iy - X - iY| = |(x_n - X) + i(y_n - Y)|$

$$\overset{\triangle}{\leq} |x_n - X| + |i(y_n - Y)| = |x_n - x| + \underbrace{|i|}_{1}|y_n - y|$$

$$= |x_n - x| + |y_n - Y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ Thus, $\lim_{n\to\infty} z_n = L$.

proof:

$(\Rightarrow)$ Suppose $\lim_{n\to\infty} z_n = L = X + iY$.

$$z_n - L = x_n + iy_n - X - iY$$
$$= (x_n - X) + i(y_n - Y)$$

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} z_n = L$ there exists $N > 0$ where $n \geq N$ then $|z_n - L| < \varepsilon$.

Thus if $n \geq N$, then

$$|x_n - X| = |\mathrm{Re}(z_n - L)| \leq |z_n - L| < \varepsilon.$$

$|\mathrm{Re}(w)| \leq |w|$

So, if $n \geq N$, then $|x_n - x| < \varepsilon$. So, $\lim_{n\to\infty} x_n = X$.

$|\mathrm{Im}(w)| \leq |w|$

Similarly, if $n \geq N$ then $|y_n - y| = |\mathrm{Im}(z_n - L)| \leq |z_n - L| < \varepsilon$.

So if $n \geq N$, then $|y_n - Y| < \varepsilon$. So, $\lim_{n\to\infty} y_n = Y$

$\square$