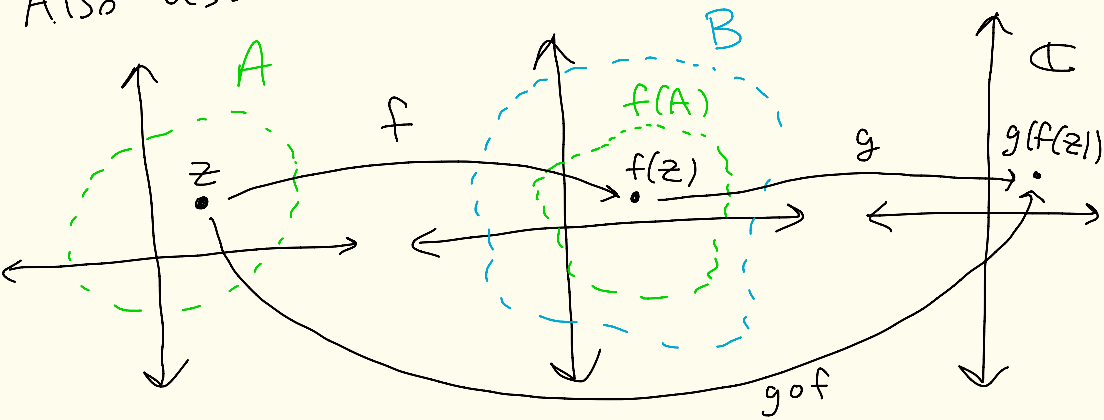
Theorem (Chain rule)

Let $A, B \subseteq \mathbb{C}$ be open sets.
Let $f : A \to \mathbb{C}$ be analytic on A
and $g : B \to \mathbb{C}$ be analytic on B.
Also assume $f(A) \subseteq B$.

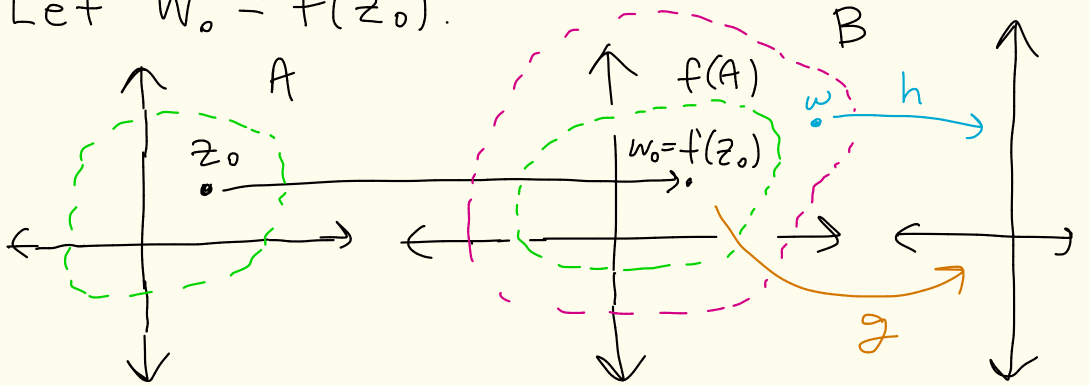


Then $g \circ f : A \to \mathbb{C}$ is analytic on A
and $(g \circ f)'(z) = g'(f(z)) f'(z)$.

proof: Let $z_0 \in A$.

We will look at the derivative at z_0.

Let $w_0 = f(z_0)$.



Define

$$h(w) = \begin{cases} \dfrac{g(w)-g(w_0)}{w-w_0} - g'(w_0) & \text{if } w \neq w_0 \\ 0 & \text{if } w = w_0. \end{cases}$$

for all $w \in B$.

Note that h is continuous on all of B. (why?) If $w \neq w_0$, since g is continuous on B, so is $\dfrac{g(w)-g(w_0)}{w-w_0} - g'(w_0)$.

What about at $w = w_0$? We have

$$\lim_{w \to w_0} h(w) = \lim_{w \to w_0} \left[\underbrace{\frac{g(w) - g(w_0)}{w - w_0}}_{\text{limits to } g'(w_0)} - g'(w_0) \right]$$

$$= g'(w_0) - g'(w_0) = 0 = h(w_0).$$

So, h is continuous at w_0.

So,

$$\lim_{z \to z_0} h(f(z)) = h(f(z_0))$$

$$\boxed{\begin{array}{l} h \text{ is cts at } w_0 = f(z_0) \\ f \text{ is cts at } z_0 \\ h \circ f \text{ is cts at } z_0 \end{array}}$$

$$= h(w_0) = 0.$$

If $f(z) \neq w_0$ $(z \in A)$, then

$$g(f(z)) - g(w_0)$$

$$= \Bigg[\underbrace{\frac{g(f(z)) - g(w_0)}{f(z) - w_0} - g'(w_0)}_{h(f(z)) \text{ when } f(z) \neq w_0} + g'(w_0) \Bigg] \Big[f(z) - w_0 \Big]$$

$$= \Big[h(f(z)) + g'(w_0) \Big] \Big[f(z) - w_0 \Big].$$

If $f(z) = w_0$ $(z \in A)$, then

$$\Big[h(f(z)) + g'(w_0) \Big] \Big[\underbrace{f(z) - w_0}_{0} \Big]$$

$$= 0 = g(w_0) - g(w_0)$$

$$= g(f(z)) - g(w_0).$$

So, $$g(f(z)) - g(w_0) = \Big[h(f(z)) + g'(w_0) \Big] \Big[f(z) - w_0 \Big]$$ for all $z \in A$.

Thus,

$$\lim_{z \to z_0} \frac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{g(f(z)) - g(w_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{\left[h(f(z)) + g'(w_0)\right]\left[f(z) - \overset{f(z_0)}{w_0}\right]}{z - z_0}$$

$$= \lim_{z \to z_0} \left[h(f(z)) + g'(w_0)\right]\left(\frac{f(z) - f(z_0)}{z - z_0}\right)$$

$$= \left[\underbrace{h(f(z_0))}_{0} + g'(w_0)\right] \cdot f'(z_0)$$

$$= g'(w_0)\, f'(z_0) = g'(f(z_0))\, f'(z_0).$$

$\blacksquare$