# HW 5 Material - Analytic functions

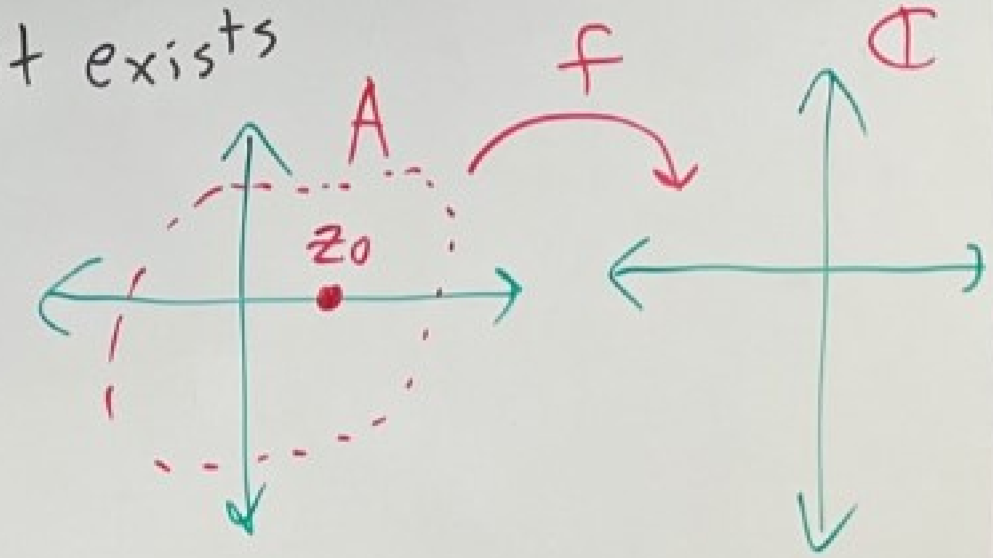
Another name for analytic is holomorphic

**Def:** Let $A \subseteq \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$

① $f$ is said to be <u>differentiable</u> at $z_0 \in A$ if

$$\boxed{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

exists. If the limit exists then we denote it by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.




② The function $f$ is said to be <u>analytic</u> on $A$ if $f$ is differentiable at each point in $A$.

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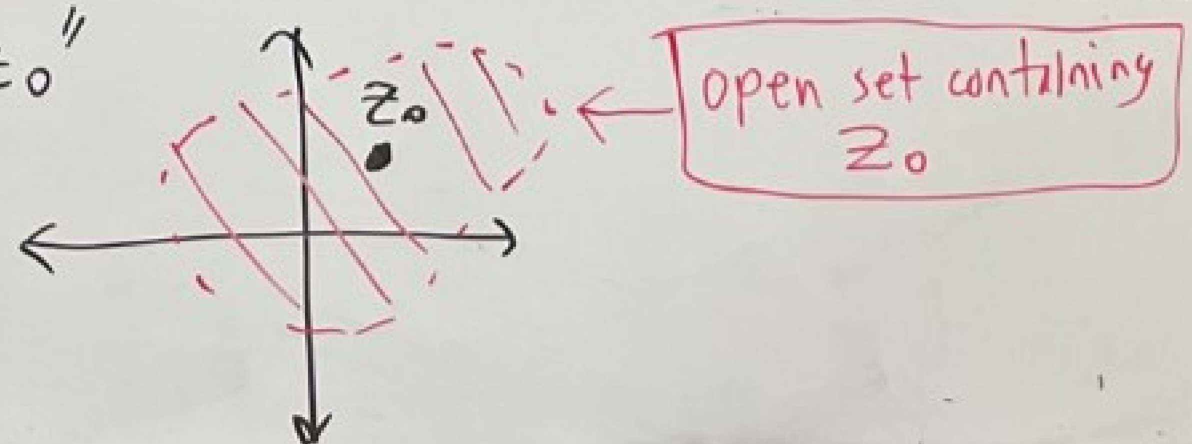
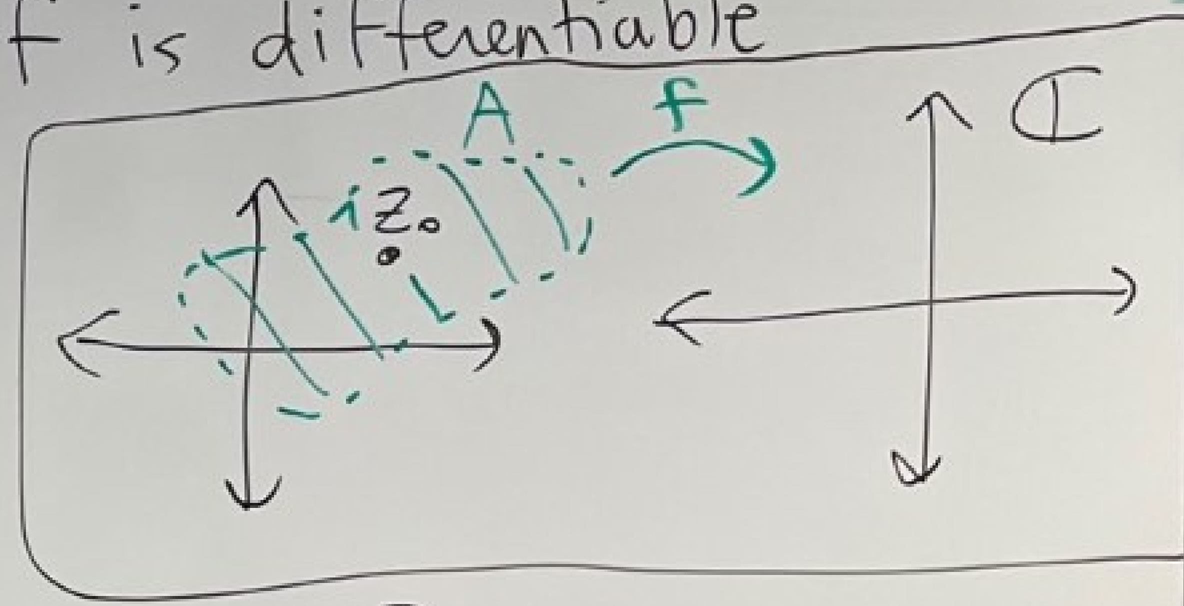
If someone says "Let $g$ be analytic at $z_0$" then they mean that $g$ is differentiable on some open set containing $z_0$.



**Theorem:** Let $A \subseteq \mathbb{C}$ where $A$ is an open set. Let $f: A \to \mathbb{C}$. Let $z_0 \in A$. If $f$ is differentiable at $z_0$, then $f$ is continuous at $z_0$.




**proof:** We are given that

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \text{ exists.}$$

We have that

$$\left(\lim_{z\to z_0} f(z)\right) - f(z_0) = \lim_{z\to z_0}\left[f(z)-f(z_0)\right] =$$

$$= \lim_{z\to z_0} \overbrace{\frac{f(z)-f(z_0)}{\underbrace{(z-z_0)}_{\text{this is ok since } z\neq z_0 \text{ in a limit}}}}^{\to f'(z_0)} \overbrace{(z-z_0)}^{\to 0} = f'(z)\cdot 0 = 0$$

Thus, $\lim\limits_{z\to z_0} f(z)-f(z_0)=0$.

So, $\lim\limits_{z\to z_0} f(z)=f(z_0)$. So, $f$ is continuous at $z_0$ $\blacksquare$

**Theorem:** Suppose that $f: A \to \mathbb{C}$ and $g: A \to \mathbb{C}$ are analytic on an open set $A \subseteq \mathbb{C}$. Then:

1. Let $\alpha, \beta \in \mathbb{C}$. Then, $\alpha f + \beta g$ is analytic on $A$ and $(\alpha f + \beta g)'(z) = \alpha f'(z) + \beta g'(z)$ for any $z \in A$.
2. $fg$ is analytic on $A$ and $(fg)'(z) = f'(z) g(z) + f(z) g'(z)$ for any $z \in A$.
3. If $g(z) \neq 0$ for all $z \in A$, then $\frac{f}{g}$ is analytic on $A$ and
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z) g(z) - f(z) g'(z)}{[g(z)]^2} \text{ for all } z \in A.$$
4. Any polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is analytic on all of $\mathbb{C}$ and $p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$ for all $z \in \mathbb{C}$
5. Let $h(z) = \frac{p_1(z)}{p_2(z)}$ be a rational function, that is $p_1$ and $p_2$ are polynomials. Then, $h$ is analytic on $\{z \in \mathbb{C} \mid p_2(z) \neq 0\} = \mathbb{C} - \{r_1, r_2, \ldots, r_n\}$ where $r_1, r_2, \ldots, r_n$ are the roots of $p_2(z)$.

proof: Let's prove ② and ④.

② Let $z_0 \in A$.

Since $f$ is analytic at $z_0$, we know $f$ is continuous at $z_0$, so $\lim_{z\to z_0} f(z) = f(z_0)$.

We have

$$\lim_{z\to z_0}\frac{(fg)(z)-(fg)(z_0)}{z-z_0} = \lim_{z\to z_0}\frac{f(z)g(z)-f(z_0)g(z_0)}{z-z_0}$$

$$= \lim_{z\to z_0}\frac{f(z)g(z)\overbrace{-f(z)g(z_0)+f(z)g(z_0)}^{0}-f(z_0)g(z_0)}{z-z_0}$$

$$= \lim_{z\to z_0}\left[\underbrace{f(z)}_{\to f(z_0)}\underbrace{\left[\frac{g(z)-g(z_0)}{z-z_0}\right]}_{\to g'(z_0)} + g(z_0)\underbrace{\left[\frac{f(z)-f(z_0)}{z-z_0}\right]}_{\to f'(z_0)}\right]$$

$$= f(z_0)g'(z_0)+g(z_0)f'(z_0) = f'(z_0)g(z_0)+f(z_0)g'(z_0) \qquad \boxed{2}$$

proof of (4)

Claim 1: $\frac{d}{dz} c = 0$ where $c \in \mathbb{C}$

pf of claim 1: Let $k(z) = c$ for all $z \in \mathbb{C}$. Then, $k'(z_0) = \lim_{z \to z_0} \frac{k(z) - k(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{c - c}{z - z_0}$

$= \lim_{z \to z_0} \frac{0}{z - z_0} = 0$ $\boxed{\text{Claim 1}}$

Claim 2: $\frac{d}{dz} z^n = n z^{n-1}$ when $n \geq 1$.

proof of claim 2: Induct on $n$.

base case: $\left(\frac{d}{dz} z\right)\Big|_{z=z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} (1) = 1.$

Assume $\frac{d}{dz} z^k = k z^{k-1}$ for some $k \geq 1$.

Then, $\frac{d}{dz} z^{k+1} = \frac{d}{dz} (z^k)(z) \overset{(2)}{=} (k z^{k-1}) z + (z^k)(1) = k z^k + z^k = (k+1) z^k$

So by induction we have claim 2 is true $\boxed{\text{claim 2}}$

proof of (4) continued...

Thus,

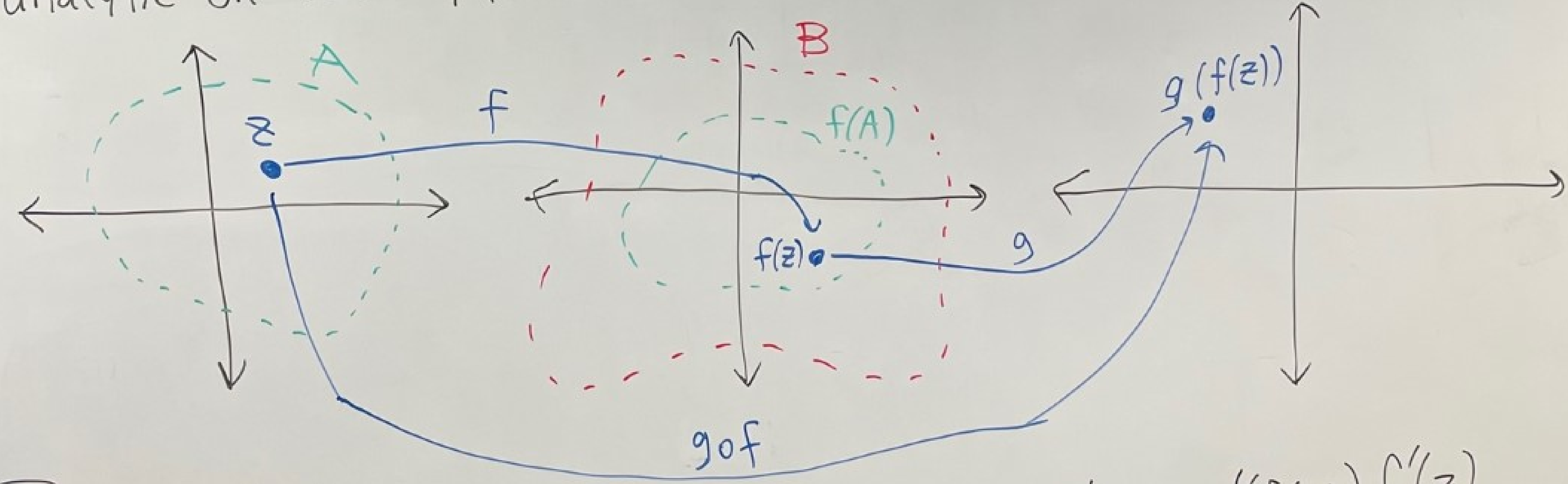
$$\frac{d}{dz}\left(a_0+a_1 z+a_2 z^2+\dots+a_n z^n\right) \overset{(1)}{=} \frac{d}{dz}a_0 + a_1\left(\frac{d}{dz}z\right) + a_2\left(\frac{d}{dz}z^2\right) + \dots + a_n\left(\frac{d}{dz}z^n\right)$$

$$\underset{\text{Claim 1 \& 2}}{=} 0 + a_1 + a_2(2z) + \dots + a_n\left(nz^{n-1}\right)$$

$$= a_1 + 2a_2 z + \dots + n a_n z^{n-1}$$

$$\boxed{(4)}$$

**Theorem (Chain rule)** Let $A, B \subseteq \mathbb{C}$ be open sets. Let $f : A \to \mathbb{C}$ be analytic on $A$ and $g : B \to \mathbb{C}$ be analytic on $B$. Assume $f(A) \subseteq B$.




Then, $g \circ f : A \to \mathbb{C}$ is analytic on $A$ and $(g \circ f)'(z) = g'(f(z))\, f'(z)$ for all $z \in A$

**Proof:** I'll post on website under todays notes. $\boxed{\phantom{x}}$