Division

To simplify $\frac{z}{w}$ where $z, w \in \mathbb{C}$ and $w \neq 0$ then multiply by $\frac{\overline{w}}{\overline{w}}$.

Ex:

$$\frac{2-5i}{1-3i} = \underbrace{\left(\frac{2-5i}{1-3i}\right)}_{z/w} \cdot \underbrace{\left(\frac{1+3i}{1+3i}\right)}_{\overline{w}/\overline{w}} = \frac{2+6i-5i-15i^2}{1+3i-3i-9i^2}$$

$i^2 = -1$

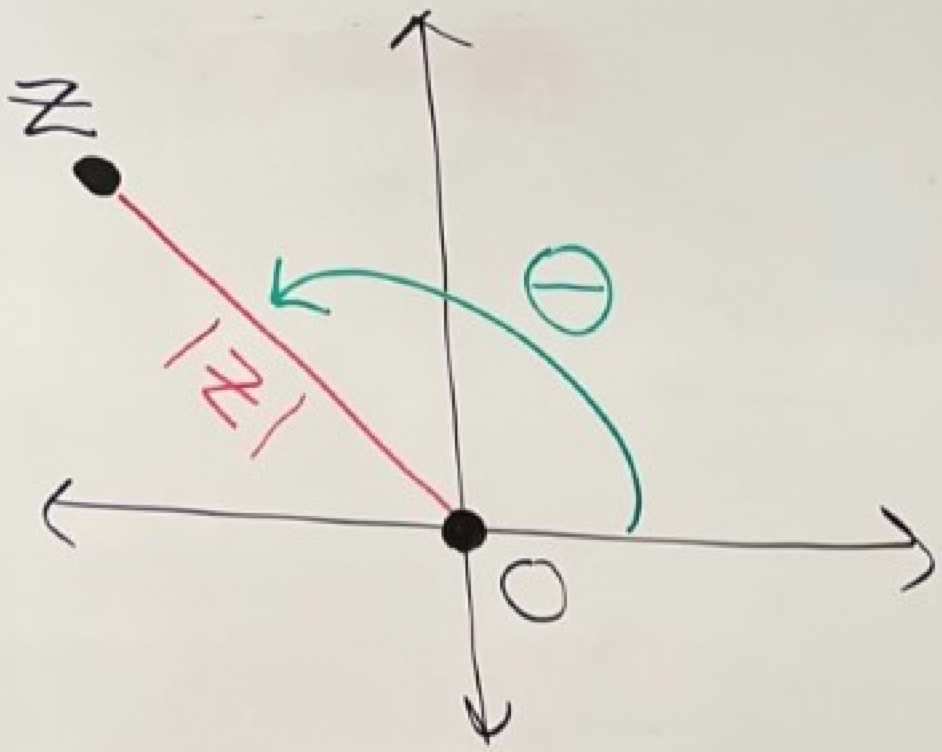
$$= \frac{17+i}{10} = \frac{17}{10} + \frac{1}{10}i$$

Idea:

$$\frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}} = \frac{z\overline{w}}{w\overline{w}}$$

and $w\overline{w}$ is a real number

Polar form of a complex number



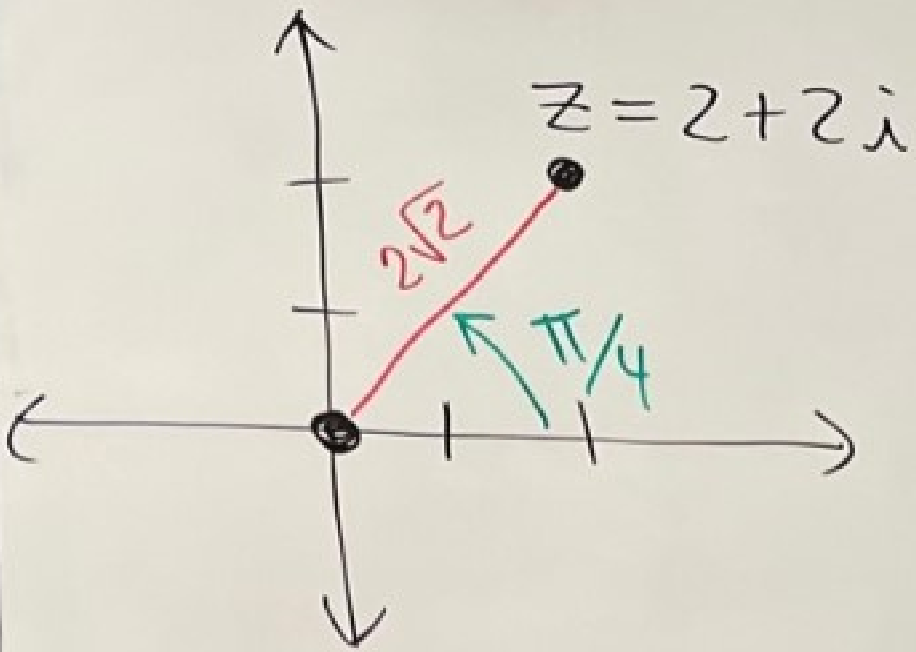
Let $r=|z|$. Consider the ray that starts at 0 and ends at z. Let θ be the angle that this ray makes with the positive x-axis. If $z=x+iy$, then by trig $x=r\cos(\theta)$ and $y=r\sin(\theta)$.

So, $z = x+iy = r\cos(\theta)+ir\sin(\theta) = \underbrace{r\left[\cos(\theta)+i\sin(\theta)\right]}_{\text{This is called the polar form of } z.}$

θ is called an <u>argument</u> of z.

We write $\theta = \arg(z)$

Ex: $z = 2 + 2i$



$$r\left[\cos(\theta) + i\sin(\theta)\right]$$

$$z = 2\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \leftarrow$$

polar form of z

$$|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Note we could have picked a different angle. So really

$$\arg(2+2i) = \frac{\pi}{4} + 2\pi k$$

where k is any integer

$\arg(z)$ is a multi-valued function

arg(z) is a multi-valued function.

We pick a $\boxed{2\pi\text{-range}}$ that makes arg(z) into a function.

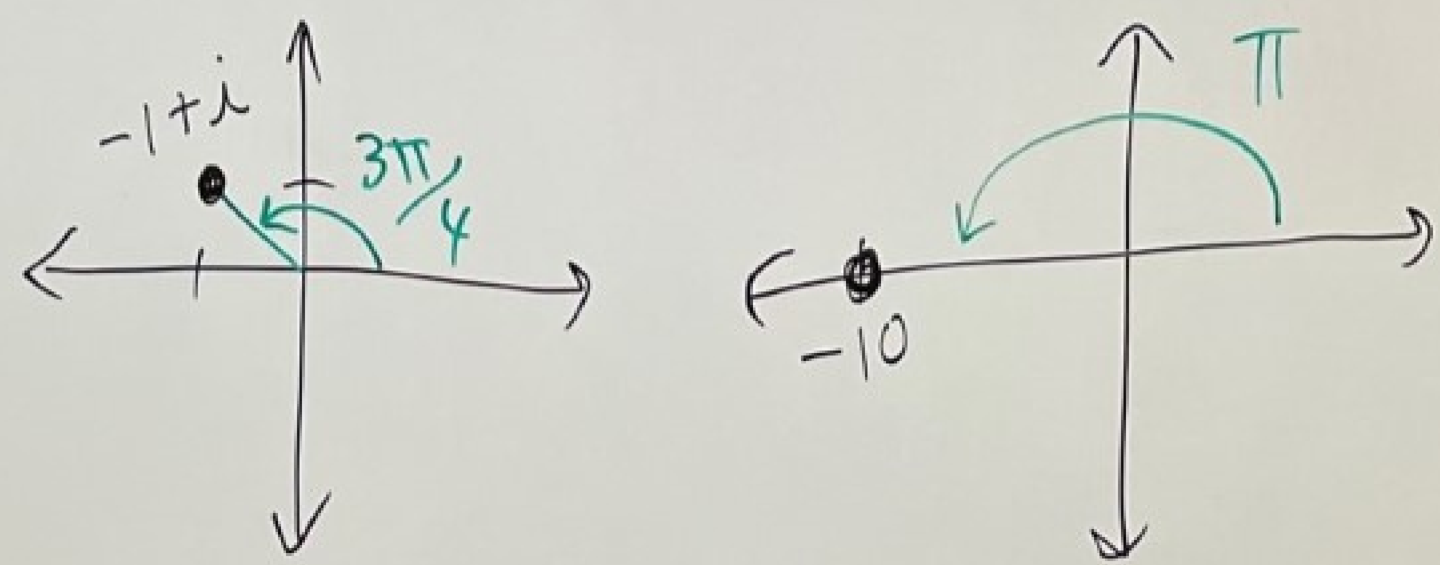
This is called choosing a <u>branch of arg</u>.

<u>Ex:</u> If we choose the branch of arg to be $[0, 2\pi)$

that is always pick $0 \le \arg(z) < 2\pi$, then

$$\arg(-1+i) = 3\pi/4$$

$$\arg(-10) = \pi$$

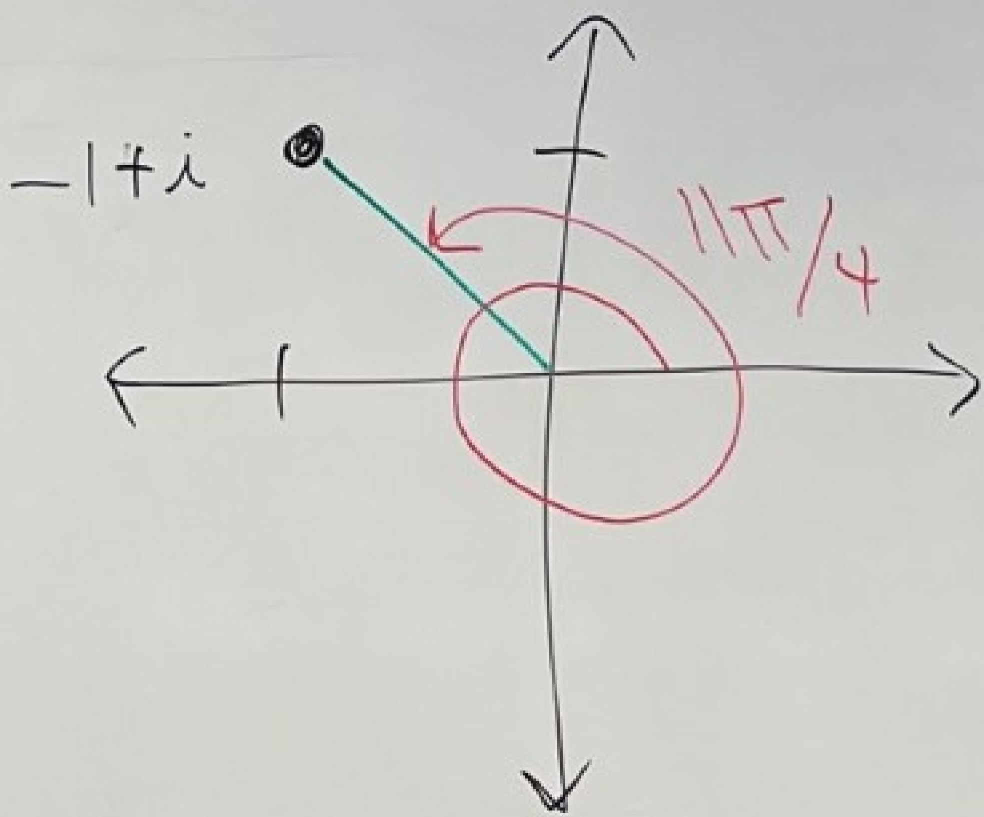


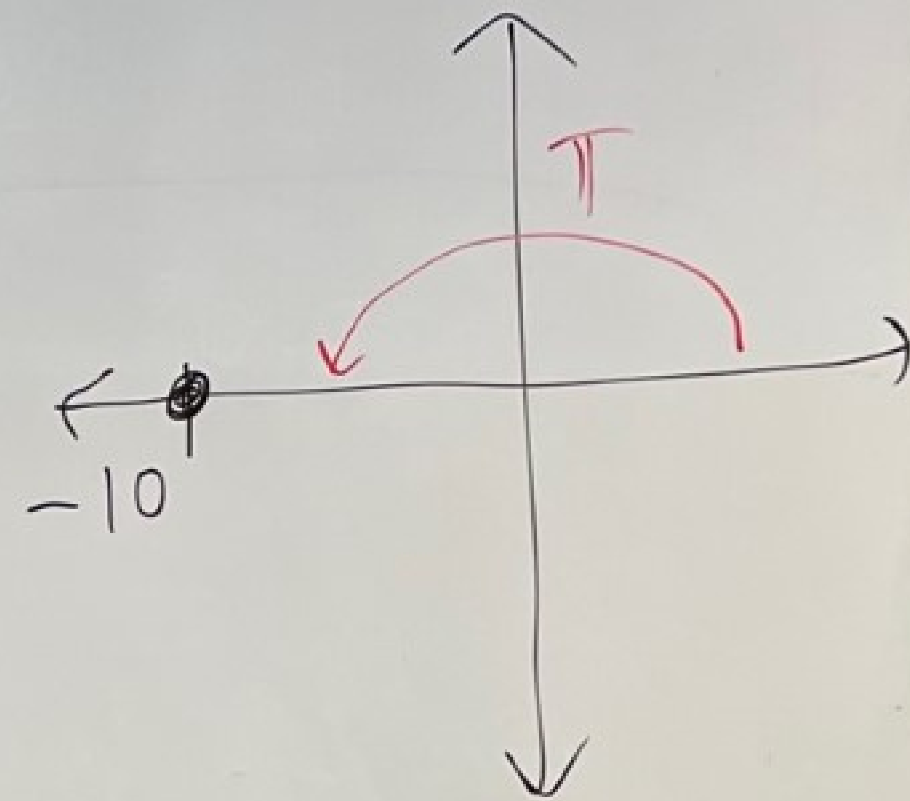
Ex: What if we choose the branch of arg to be $\left(\frac{3\pi}{4}, \frac{11\pi}{4}\right]$

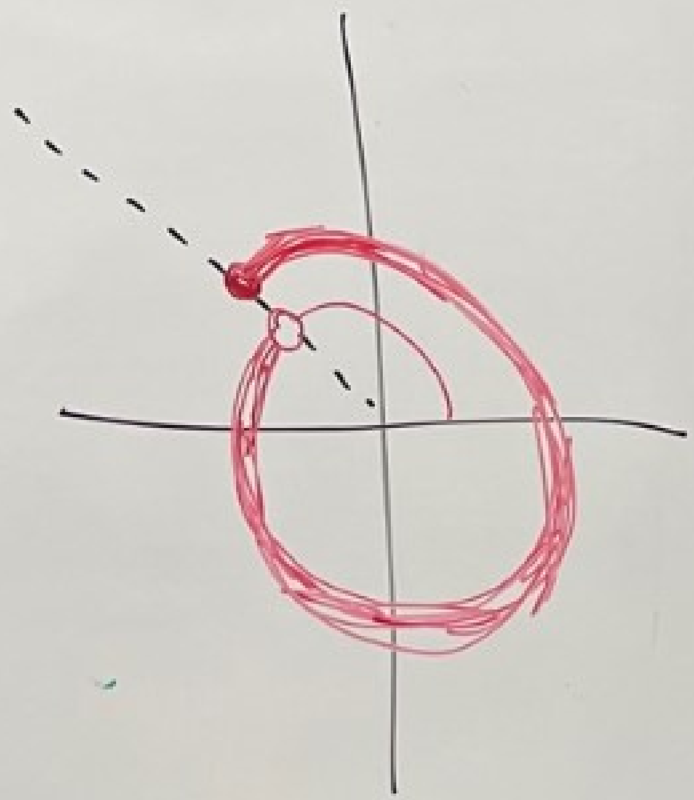
So we pick $\frac{3\pi}{4} < \arg(z) \leq \frac{11\pi}{4}$

$\arg(-1+i) = 11\pi/4$

$\arg(-10) = \pi$







Proposition: Let $z, w \in \mathbb{C}$. Then:

We say that $x+iy = a+ib$ if $x=a$ and $y=b$

1. $\overline{z+w} = \bar{z} + \bar{w}$
2. $\overline{zw} = \bar{z}\,\bar{w}$
3. $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$ if $w \neq 0$
4. $|z|^2 = z\bar{z}$
5. $z = \bar{z}$ iff z is a real number.
6. $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) = \frac{z-\bar{z}}{2i}$
7. $\bar{\bar{z}} = z$
8. $|zw| = |z||w|$
9. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ if $w \neq 0$
10. $|\bar{z}| = |z|$
11. $\text{Re}(z) \leq |\text{Re}(z)| \leq |z|$
    $\text{Im}(z) \leq |\text{Im}(z)| \leq |z|$
12. $|z+w| \leq |z| + |w|$ ← Triangle inequality
13. $|z+w| \geq \big||z| - |w|\big|$
14. $|z-w| \geq \big||z| - |w|\big|$

proof: We will prove (11) – (14).

proof of (11):

We have that

$$\mathrm{Re}(z) \le |\mathrm{Re}(z)| = \sqrt{(\mathrm{Re}(z))^2} \le \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2} = |z|$$

since $\mathrm{Re}(z)$ is a real #
$x \le |x|$
if $x \in \mathbb{R}$

$|x| = \sqrt{x^2}$
if $x \in \mathbb{R}$

$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$

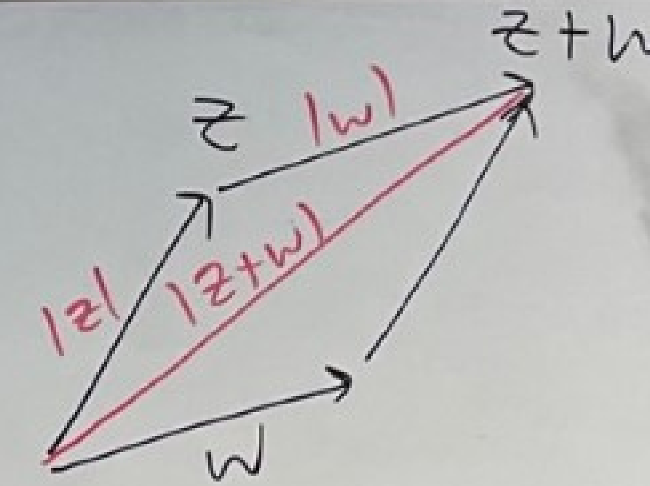
Similarly for $\mathrm{Im}(z)$.

Proof of 12

$$
\begin{aligned}
|z+w|^2 &\overset{4}{=} (z+w)\overline{(z+w)} \overset{1}{=} (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&\overset{3/7}{=} z\bar{z} + \left(z\bar{w} + \overline{z\bar{w}}\right) + w\bar{w} \\
&\overset{6}{=} z\bar{z} + 2\,\mathrm{Re}(z\bar{w}) + w\bar{w} \overset{4}{=} |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2 \\
&\overset{11}{\leq} |z|^2 + 2|z\bar{w}| + |w|^2 \overset{8}{=} |z|^2 + 2|z||\bar{w}| + |w|^2 \\
&\overset{10}{=} |z|^2 + 2|z||w| + |w|^2 = \left(|z|+|w|\right)^2
\end{aligned}
$$

Now take square roots to get $|z+w| \leq |z| + |w|$

proof of 13 $\left[\, |z+w| \geq \big||z|-|w|\big| \,\right]$



If $a,b \in \mathbb{C}$, then

$$|a| = |a+b-b| \overset{12}{\leq} |a+b| + |-b| = |a+b| + |b|$$

So, $\boxed{|a|-|b| \leq |a+b|}$ $(*)$

Now let's go back to $z,w \in \mathbb{C}$.

Case 1: Suppose $|z| \geq |w|$.

Then, $|z|-|w| \geq 0$.

So, $\big|\underbrace{|z|-|w|}_{\geq 0}\big| = |z|-|w| \overset{(*)}{\leq} |z+w|$

Case 2: Suppose $|z| < |w|$. So, $\underbrace{|z|-|w|}_{<0}$

Then,

$$\big|\underbrace{|z|-|w|}_{<0}\big| = -(|z|-|w|)$$
$$= |w|-|z| \overset{(*)}{\leq} |w+z| = |z+w|$$