proof of 4 $\quad \sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

We have

$$\sin(z)\cos(w) + \cos(z)\sin(w) = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)\left(\frac{e^{iw}+e^{-iw}}{2}\right) + \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{iw}-e^{-iw}}{2i}\right)$$

$$= \frac{1}{4i}\left[e^{i(z+w)} + e^{i(z-w)} - e^{i(w-z)} - e^{-i(z+w)} + e^{i(z+w)} - e^{i(z-w)} + e^{i(w-z)} - e^{-i(z+w)}\right]$$

$$= \frac{1}{4i}\left(2e^{i(z+w)} - 2e^{-i(z+w)}\right) = \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \sin(z+w) \quad \blacksquare$$

The logarithm — In real analysis, $\ln(y)$ is the inverse of $y=e^x$.

Let's try to do this in $\mathbb{C}$.

Suppose $z=e^w$ where $z\neq 0$.

We want to solve for w and define $\log(z)=w$.

Let $z=re^{i\theta}$ and $w=x+iy$.

Then $z=e^w$ becomes $re^{i\theta}=e^{x+iy}=e^x e^{iy}$. Recall, $|e^x e^{iy}|=e^x$

and $|re^{i\theta}|=r$.

So, $\boxed{r=e^x}$ Thus, $\boxed{e^{i\theta}=e^{iy}}$

So, $x=\ln(r)=\ln(|z|)$.

And, $y=\theta+2\pi k$ where $k=0,\pm1,\pm2,\dots$

We want $\log(z)=\ln|z|+i\arg(z)$

but it has an infinite #

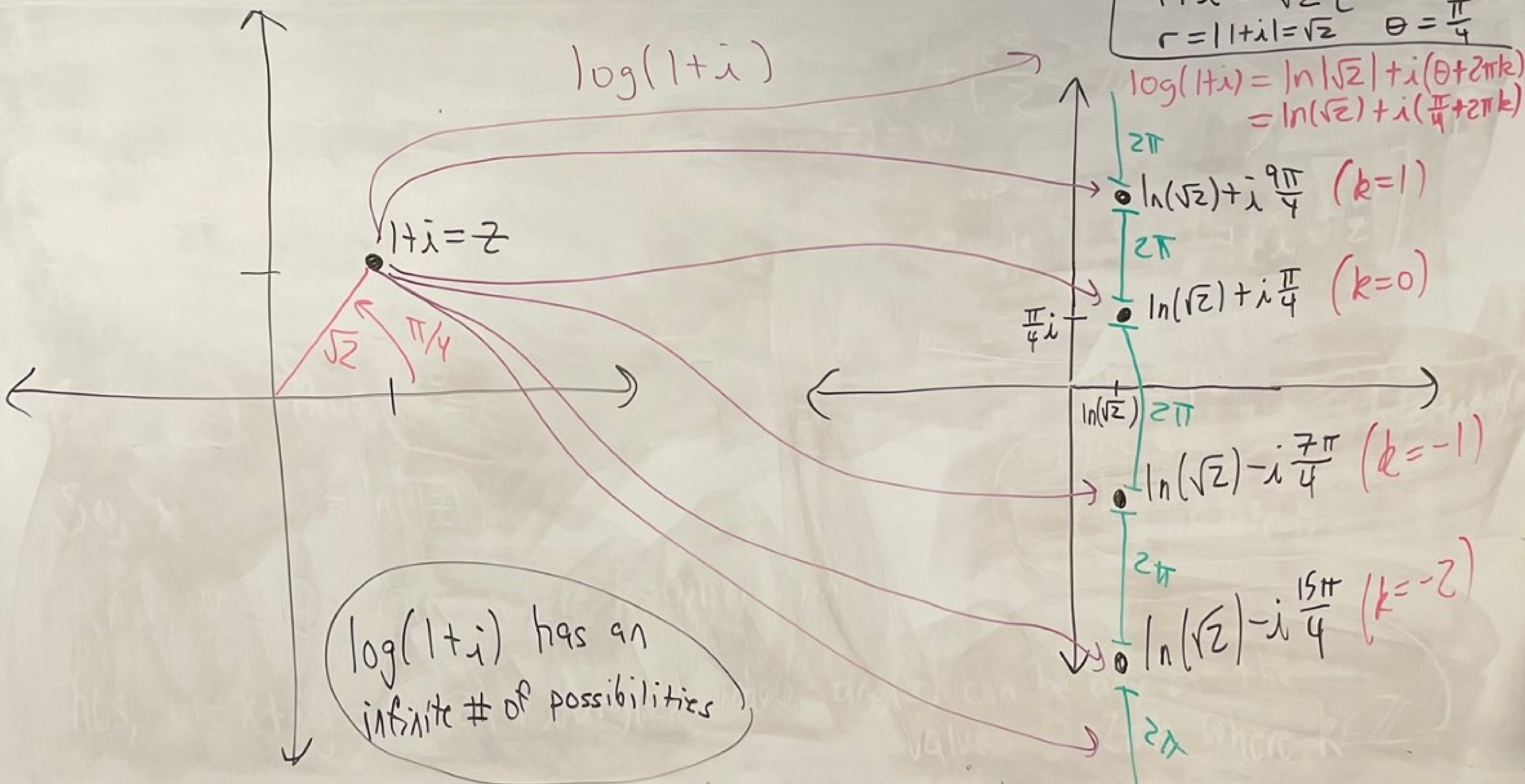
of outputs.

Thus, $w=x+iy=\ln|z|+i\arg(z)$ where $\arg(z)$ can be any of the values $\theta+2\pi k$ where $k\in\mathbb{Z}$.

Let's draw a picture for $\log(1+i)$

$$e^{w} = z = 1+i$$
$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$
$$r = |1+i| = \sqrt{2} \qquad \theta = \frac{\pi}{4}$$

$$\log(1+i) = \ln|\sqrt{2}| + i(\theta + 2\pi k)$$
$$= \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2\pi k\right)$$



log(1+i) has an infinite # of possibilities

Recall $f(z) = e^z$ is 1-1 on $A_c = \{ x+iy \mid x, y \in \mathbb{R},\ c \leq y < (c+2\pi) \}$

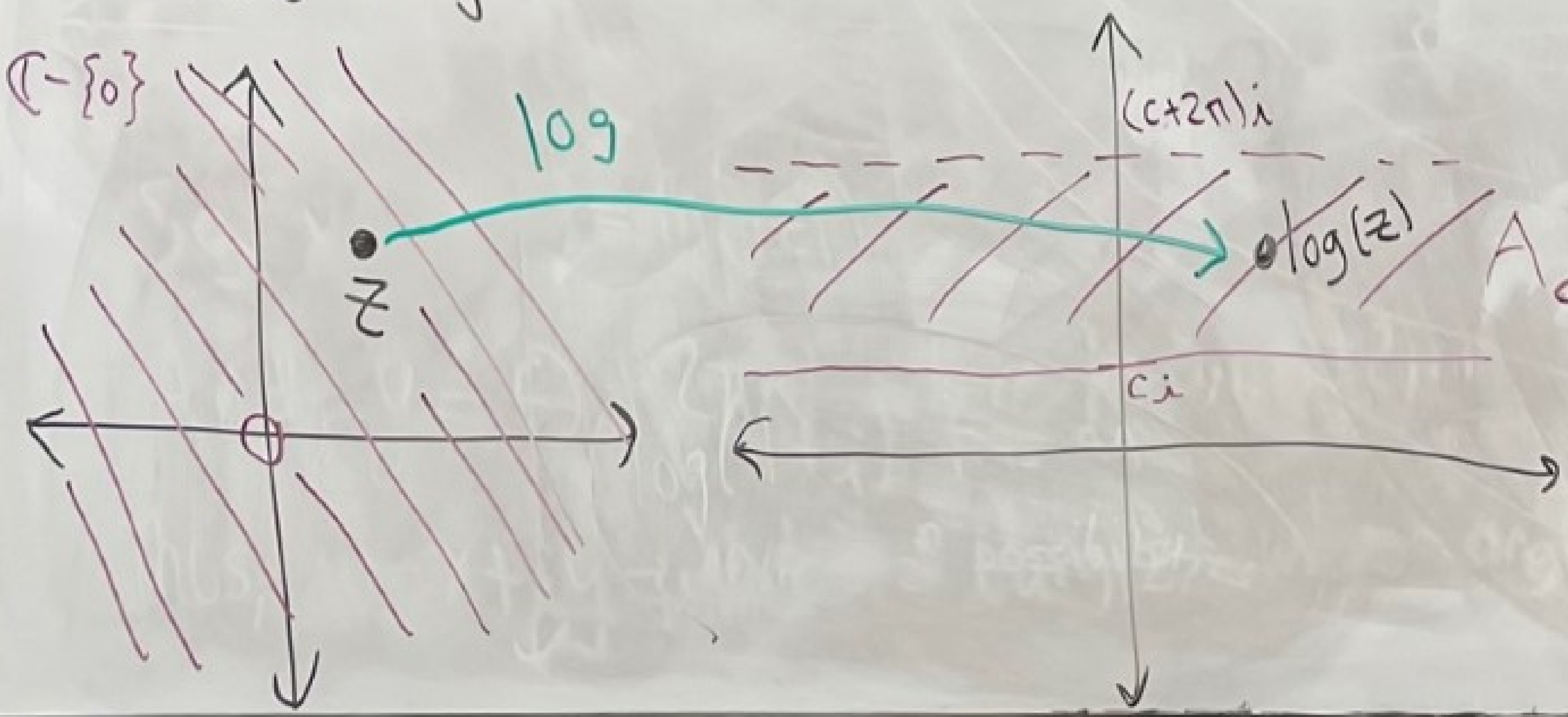
So we can pick a $c \in \mathbb{R}$ and use the above to make log into a function

Def: Let $c \in \mathbb{R}$.

Define the <u>logarithm function</u>, $\log : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$ by

$$\log(z) = \ln|z| + i \arg(z)$$

where $\arg(z)$ satisfies $c \leq \arg(z) < c + 2\pi$.

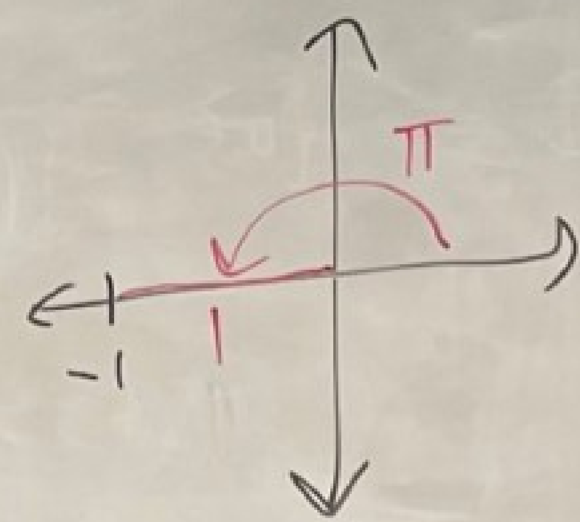


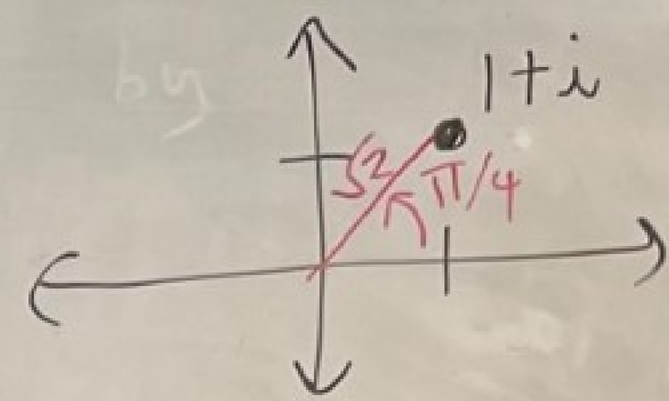
The range of this log function is A_c

This is called <u>picking a branch of the logarithm</u>

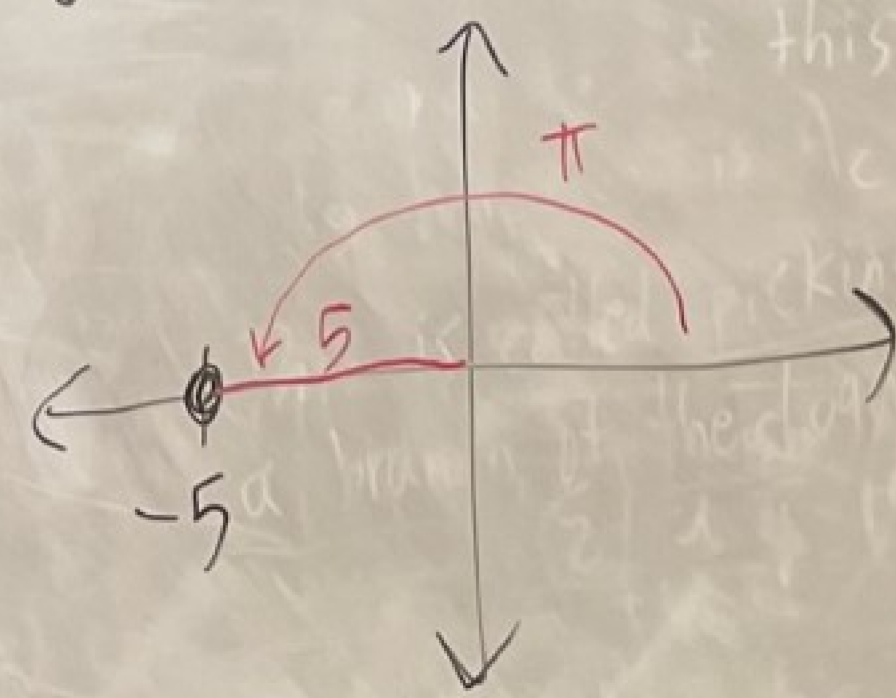
Ex: Let's pick $[0, 2\pi)$ to be our branch of log $\left[\begin{array}{l}\text{That is, } c=0 \\ 0 \le \arg(z) < 2\pi\end{array}\right]$

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln(\sqrt{2}) + i\frac{\pi}{4}$$





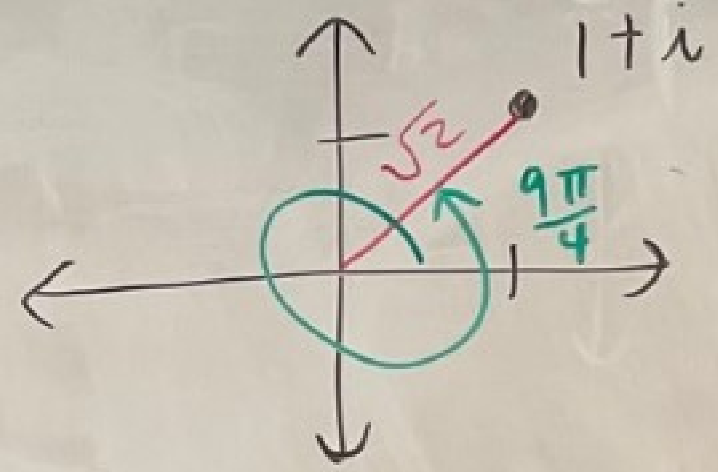
$$\log(-5) = \ln|-5| + i\arg(-5)$$
$$= \ln(5) + i\pi$$



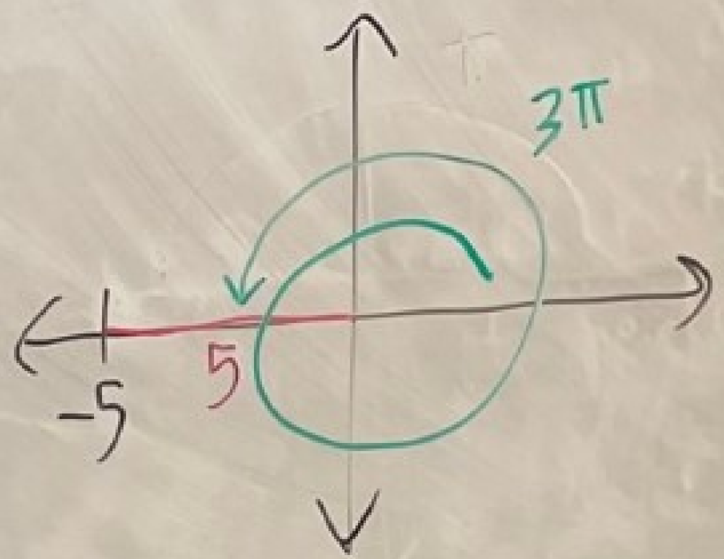
$$\log(-1) = \ln|-1| + i\arg(-1)$$
$$= \ln(1) + i\pi = 0 + i\pi = i\pi$$

Ex: Let's pick $[2\pi, 4\pi)$ to be our branch of log $\left[\begin{array}{l}\text{That is, } c = 2\pi \\ 2\pi \leq \arg(z) < 4\pi\end{array}\right]$

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln(\sqrt{2}) + i\frac{9\pi}{4}$$

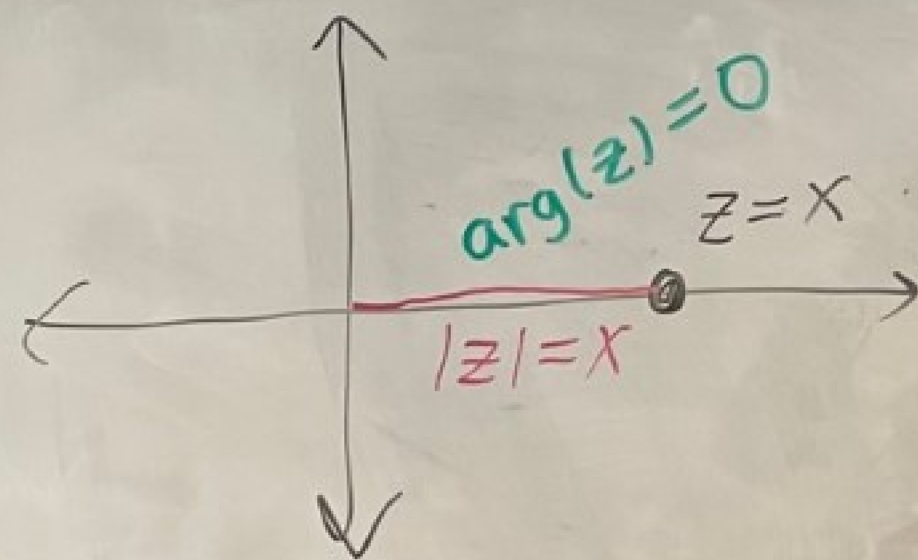


$$\log(-5) = \ln|-5| + i\arg(-5)$$
$$= \ln(5) + i3\pi$$



Note: Suppose you pick a branch of the log where the $\arg(z)$ range contains 0, for ex $[0, 2\pi)$ or $[-\pi, \pi)$.

Suppose $z = x + 0i = x$ where $x > 0$.



Then

$$\log(z) = \ln|z| + i\arg(z)$$
$$= \ln(x) + i0$$
$$= \ln(x)$$

So, $\log(z) = \log(x)$ matches up with the real analysis/calculus $\ln(x)$.

So, this extends $\ln(x)$ to all of $\mathbb{C}$.

Complex powers

Motivation: Let $a, b \in \mathbb{R}$ and $a > 0$.
Then in real analysis we have

$$a^b = e^{\ln(a^b)} = e^{b \ln(a)}$$

$$2^3 = e^{\ln(2^3)} = e^{3\ln(2)}$$

$$\ln(x) = \log_e(x)$$

Def: Let $a, b \in \mathbb{C}$, $a \neq 0$.

Define $a^b = e^{b \log(a)}$

where $\log$ is some branch of the logarithm