<u>Theorem:</u> Suppose $A \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$ and $D^*(z_0; r) \subseteq A$ for some $r > 0$. Suppose $f(z) = f(x+iy) = u(x,y) + i v(x,y)$. Let $z_0 = x_0 + i y_0$ and $w_0 = u_0 + i v_0$.

Then:

(1) $\lim\limits_{z \to z_0} f(z) = \lim\limits_{x+iy \to x_0+iy_0} f(z) = u_0 + i v_0$

complex limit

if and only if

(2) $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

(2) $\mathbb{R}^2$ limits

<u>Ex:</u>

$x+iy = 1+i$

$(x,y) = (1,1)$

$$\lim_{z \to 1+i} z^2 = \lim_{x+iy \to 1+i} \left[(x^2 - y^2) + i\, 2xy \right]$$

$$= \lim_{(x,y)\to(1,1)} \left[x^2 - y^2 \right] + i \lim_{(x,y)\to(1,1)} \left[2xy \right]$$

$$= \left[1^2 - 1^2 \right] + i \left[2(1)(1) \right] = 2i$$

proof: You can try (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1)

Let $\varepsilon > 0$.

Suppose $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

So there exist $\delta_1 > 0$ so that if

$$\underbrace{0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_1}_{0 < |(x,y)-(x_0,y_0)| < \delta_1} \quad \text{and } (x,y) \in A$$

then $|u(x,y) - u_0| < \frac{\varepsilon}{2}$.

And there exists $\delta_2 > 0$ so that if

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2 \quad \text{and } (x,y) \in A$$

then $|v(x,y) - v_0| < \frac{\varepsilon}{2}$.

Note: $\mathbb{R}^2$	$\mathbb{C}$
$\sqrt{(x-x_0)^2+(y-y_0)^2} = \lvert(x,y)-(x_0,y_0)\rvert$	$\sqrt{(x-x_0)^2+(y-y_0)^2} = \lvert z-z_0\rvert$, $z = x+iy$, $z_0 = x_0+iy_0$

So if $z \in A$ and

$$0 < \underbrace{|z - z_0|}_{\sqrt{(x-x_0)^2+(y-y_0)^2}} < \min\{\delta_1, \delta_2\}$$ then

$$|f(x,y) - (u_0 + i v_0)|$$

$$= |u(x,y) + i v(x,y) - u_0 - i v_0|$$

$$= |(u(x,y) - u_0) + i(v(x,y) - v_0)|$$

$$\leq |u(x,y) - u_0| + |i(v(x,y) - v_0)|$$

$$= |u(x,y) - u_0| + \underbrace{|i|}_{1}\,|v(x,y) - v_0|$$

$$= |u(x,y) - u_0| + |v(x,y) - v_0|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So,

$$\lim_{z \to z_0} f(z) = u_0 + i v_0$$

$\blacksquare$