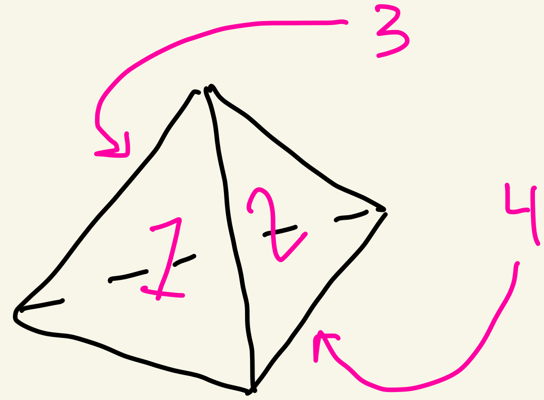
Math 4740
1/29/25

Example of making a probability space

Let's make a probability space for rolling a 4-sided die.



The "sample space" of all possible outcomes will be

$$S = \{1, 2, 3, 4\}$$

The set of "events" in this case will be the set of all subsets of S.

Omega

$$\Omega = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\},$$
$$\{1,2\}, \{1,3\}, \{1,4\},$$

$\{2,3\}, \{2,4\}, \{3,4\},$
$\{1,2,3\}, \{2,3,4\}, \{1,3,4\},$
$\{1,2,4\}, \{1,2,3,4\} \}$

Ω represents all the sets we want to be able to measure the probability of.

What do the "events" in Ω represent?

$\{2\} \leftarrow$ represents a 2 occuring when you roll the die

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$\{1,4\} \leftarrow$ represents either a 1 or a 4 occured when you rolled the die

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$\{1,3,4\} \leftarrow$ represents either 1 or 3 or 4 occured when you rolled the die

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$\{1,2,3,4\} \leftarrow$ represents either 1 or 2 or 3 or 4 occured

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$\emptyset \longleftarrow$ represents no number occured. This can't happen

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Now we make a probability function $P : \Omega \longrightarrow \mathbb{R}$

input to P

output

Let's assume each side of the die is equally likely.

First define

$$P(\phi) = 0$$

Now assign the values of P on the events of size 1.

$$\left.\begin{aligned} P(\{1\}) &= 1/4 \\ P(\{2\}) &= 1/4 \\ P(\{3\}) &= 1/4 \\ P(\{4\}) &= 1/4 \end{aligned}\right\} \text{note this adds up to } 1$$

Now extend P to all other events by doing disjoint sums.

For example, define:

$$P(\{1,4\}) = P(\{1\}) + P(\{4\})$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(\{1,2,3\}) = P(\{1\}) + P(\{2\}) + P(\{3\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

$$P(S) = P(\{1,2,3,4\})$$
$$= P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Def: A probability space consists of two sets and a function (S, Ω, P).

S is called the sample space of the experiment. The elements of S are called the outcomes of the experiment.

Ω is a set whose elements are subsets of S.

The elements of Ω are called events.

$P : \Omega \to \mathbb{R}$ is a function that assigns to each E in Ω a probability $P(E)$.

Furthermore, the following

properties must hold:

1. S is an element of Ω — so you can measure $P(S)$

2. If E is in Ω, then $\overline{E}$ is in Ω.

3. If $E_1, E_2, E_3, \ldots$ are in Ω, then $\bigcup_i E_i$ is in Ω

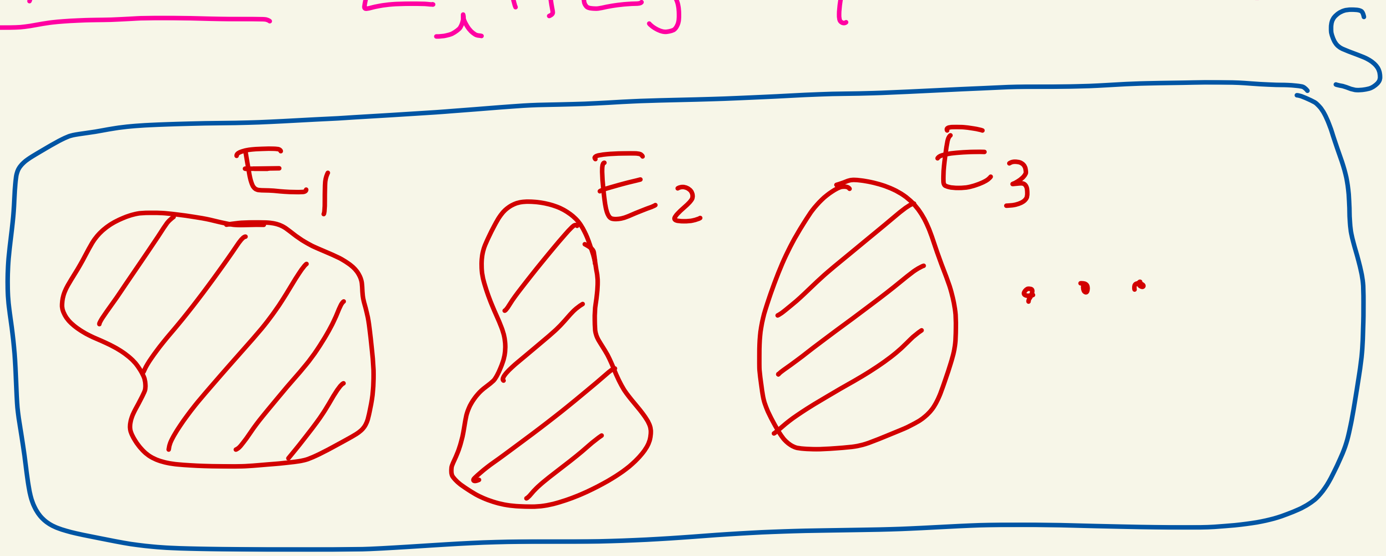
4. $0 \leq P(E) \leq 1$ for all E in Ω

5. $P(S) = 1$

6. If $E_1, E_2, E_3, \ldots$ is a finite or infinite

sequence of events from Ω
that are pair-wise disjoint

<u>means:</u> $E_i \cap E_j = \phi$ if $i \neq j$



then

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

$$P(E_1 \cup E_2 \cup E_3 \cup \cdots) = P(E_1) + P(E_2) + P(E_3) + \cdots$$

END OF DEF

I think this def is based on the work of Andrey Kolmogorov 1930s

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<u>Remark:</u> A set Ω satisfying (1), (2), (3) above is called a <u>σ-algebra</u> or <u>σ-field</u>

How to define a probability space when S is finite

Suppose S is a finite sample space.
Define Ω to be the set of <u>all</u> subsets of S.

For each outcome ω in S pick a real number n_ω with $0 \le n_\omega \le 1$ and define $P(\{\omega\}) = n_\omega$

<u>Ex:</u> $S = \{1, 2, 3, 4\}$

$P(\{1\}) = \frac{1}{4} = n_1$

$P(\{2\}) = \frac{1}{4} = n_2$

$P(\{3\}) = \frac{1}{4} = n_3$

$P(\{4\}) = \frac{1}{4} = n_4$

At the same time pick the numbers so that

$$\underbrace{\sum_{w \in S}}_{\text{means sum over all } w \text{ in } S} n_w = 1$$

In our example
$n_1 + n_2 + n_3 + n_4 = 1$

Now extend P to all of Ω as follows:

If $E = \{w_1, w_2, \ldots, w_r\}$ is an event, define

$$P(E) = \sum_{i=1}^{r} P(\{w_i\})$$

$$= P(\{w_1\}) + P(\{w_2\}) + \cdots + P(\{w_r\})$$

$$\boxed{\text{Ex: } P(\{1,2\}) = P(\{1\}) + P(\{2\})}$$

If $E = \phi$, define

$$P(\phi) = 0.$$

Theorem: The above construction creates a probability space

proof: See online notes