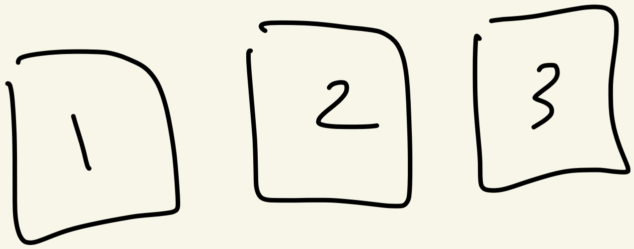
Math 4740
2/24/25

Topic 3 – Conditional Probability

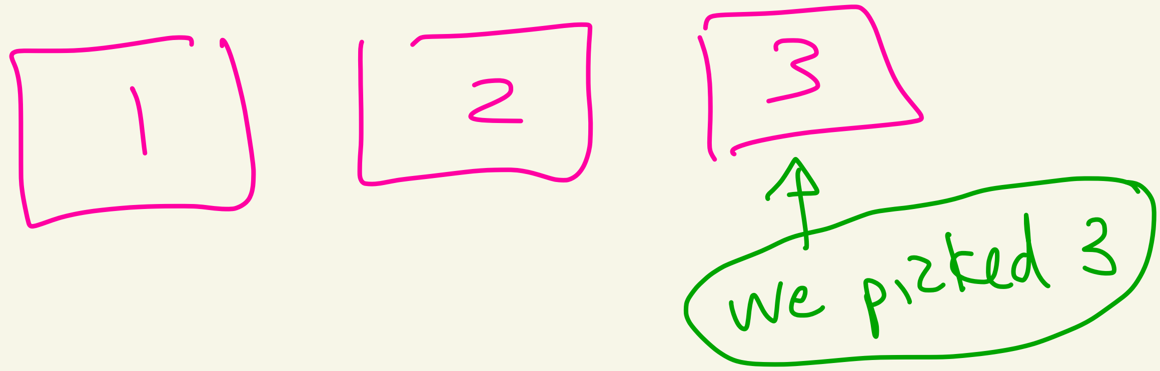
Monty Hall problem

Three doors 

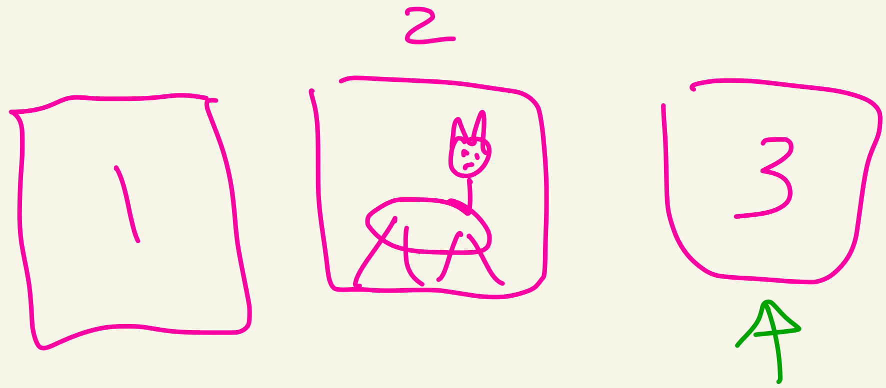
behind them: 1 car, 2 goats

You pick a door.
Monty Hall reveals a different door with a goat. Then asks you: Do you want to stay with your original choice or switch doors?
You're stuck with your choice after this point.

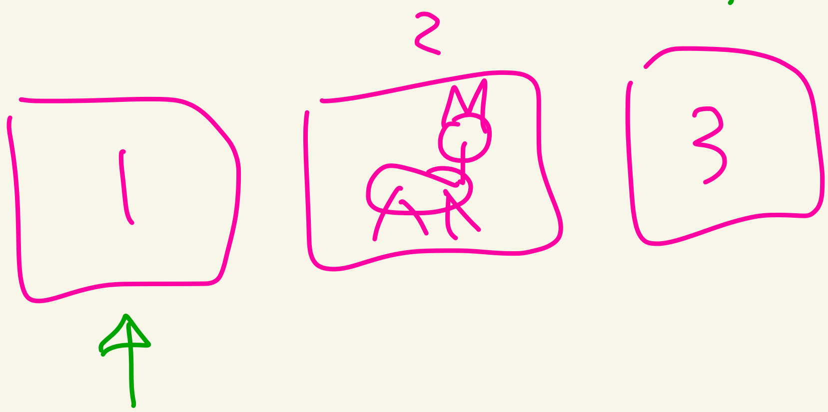
Ex:



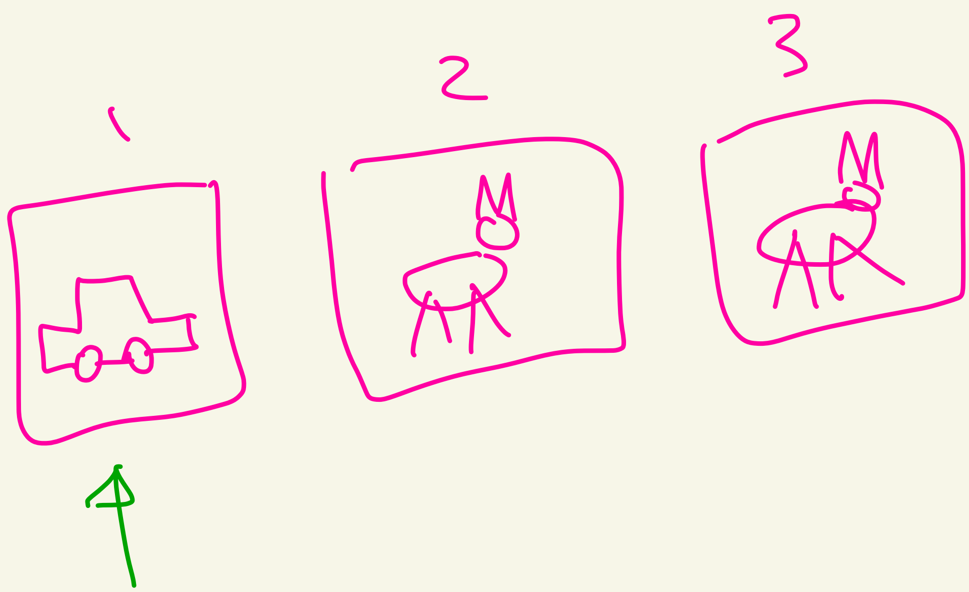
MH revealed:



we switched to 1



we won the car!



Analysis of our idea

Suppose we pick door 1 and once MH reveals a goat we switch doors. What's the chances we win the car with this strategy?

Table of possibilities

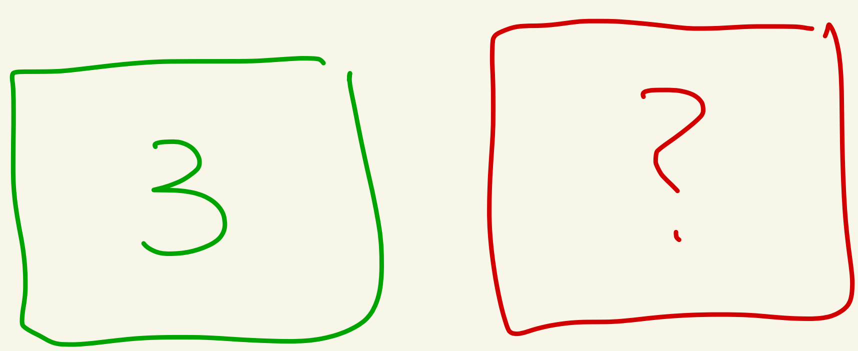
door 1	door 2	door 3	switch from door 1	stay with door 1
car	goat	goat	LOSE	WIN
goat	car	goat	WIN	LOSE
goat	goat	car	WIN	LOSE

WIN 2/3 OF THE TIME

Now on to conditional probability.

<u>Ex:</u> Suppose we roll two 6-sided dice, a green die and red die.

Suppose the green die stops rolling and lands on a 3, but the red die is still rolling. What's the probability that the sum of the dice is 8?



Starting sample space → new sample space when green die stops on 3

S

(1,1)
(1,2)
(1,3)
⋮
(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)
⋮
(5,6)
(6,6)

→

S'

(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)

36 elements
(green, red)

6 outcomes

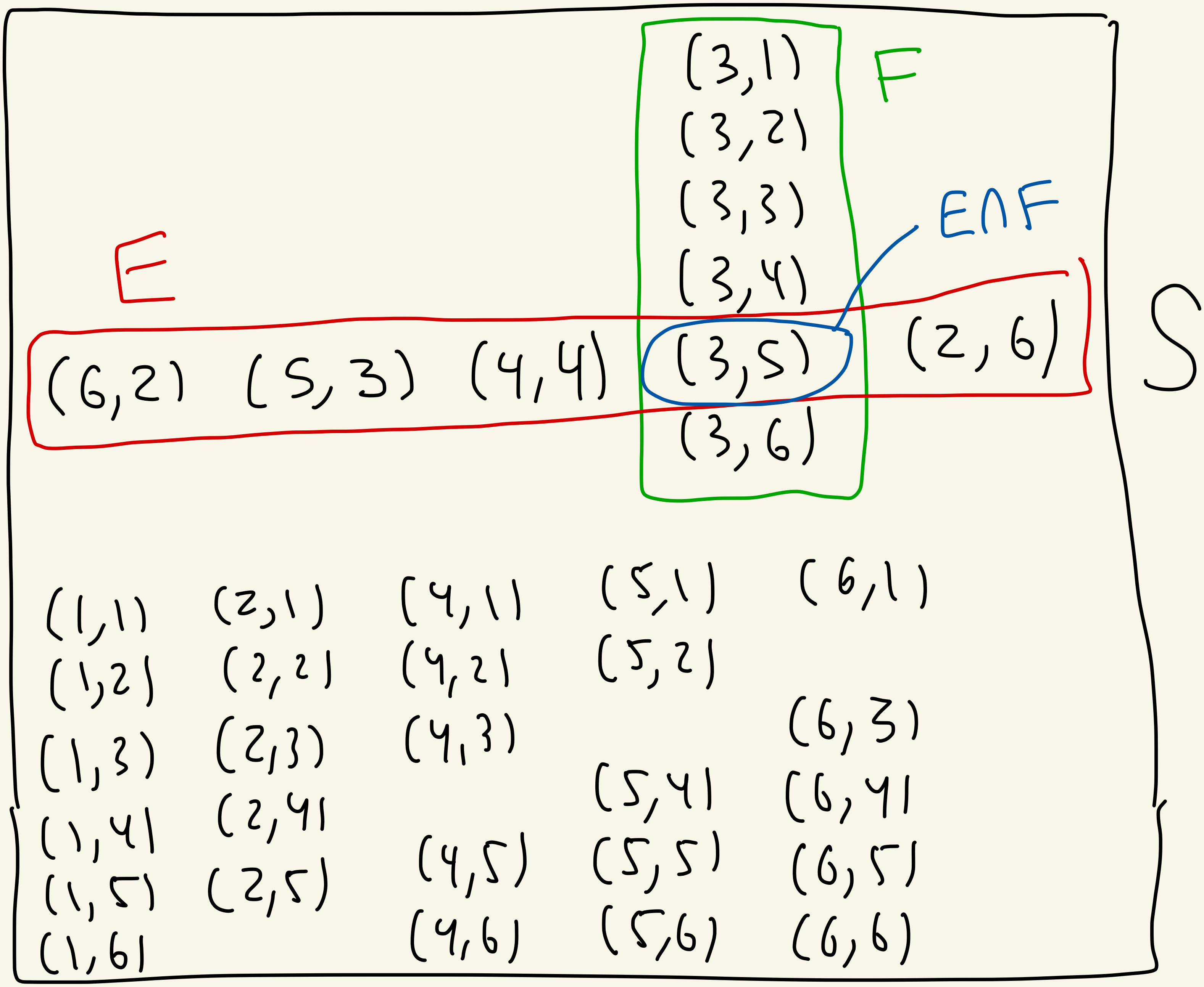
probability is 1/6

Let's make a formula for this without having to "shrink" the sample space S and also we want a formula that we can use when the outcomes are not equally likely.

Let E be the event in S where the sum of the dice is 8,

Let F be the event in S where the green die is 3.

We want the "conditional probability" of E occuring given that F "already occured."



$$\underbrace{\frac{|E\cap F|}{|F|}}_{\text{we calculated this and got } 1/6} = \frac{|E\cap F|/|S|}{|F|/|S|} \overset{\text{because outcomes equally likely}}{=} \frac{P(E\cap F)}{P(F)} = \frac{1/36}{6/36} \sim 1/6$$

Def: Let (S, Ω, P) be a probability space. Let E and F be two events with $P(F) > 0$. Define the **conditional probability** that E occurs given that F occured as:

$$\underbrace{P(E|F)}_{\substack{\text{notation}\\ \text{of conditional}\\ \text{probability}\\ \text{of } E \text{ given } F}} = \frac{P(E \cap F)}{P(F)}$$

Ex: Suppose you roll two 8 sided dice. You can't see the outcome, but your friend can. They tell you that the sum of the dice is divisible by 5. Given this, what's the probability that both dice are equal to 5?

$$S = \left\{ (a,b) \;\middle|\; \begin{matrix} a = 1, 2, \ldots, 8 \\ b = 1, 2, \ldots, 8 \end{matrix} \right\}$$

$$|S| = 8 \cdot 8 = 64$$

Let F be the event that the sum of the dice is divisible by 5. And E is both dice are equal to 5.

Want $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$

$E = \{(5,5)\}$

$F = \{(1,4), (2,3), (3,2), (4,1),$ sum 5
$(5,5), (4,6), (6,4), (2,8),$ sum 10
$(8,2), (3,7), (7,3),$
$(7,8), (8,7)\}$ sum 15

$E \cap F = \{(5,5)\}$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/64}{13/64}$$

$$= \boxed{1/13} \approx \boxed{7.7\%}$$