Math 4740

2/26/25

<u>Theorem:</u> Let (S, Ω, P) be a probability space.

① Let A and B be events with $P(A) > 0$. Then

$$P(A \cap B) = P(A) \cdot P(B|A)$$

② See notes online for formula $P(A_1 \cap A_2 \cap \dots \cap A_n)$

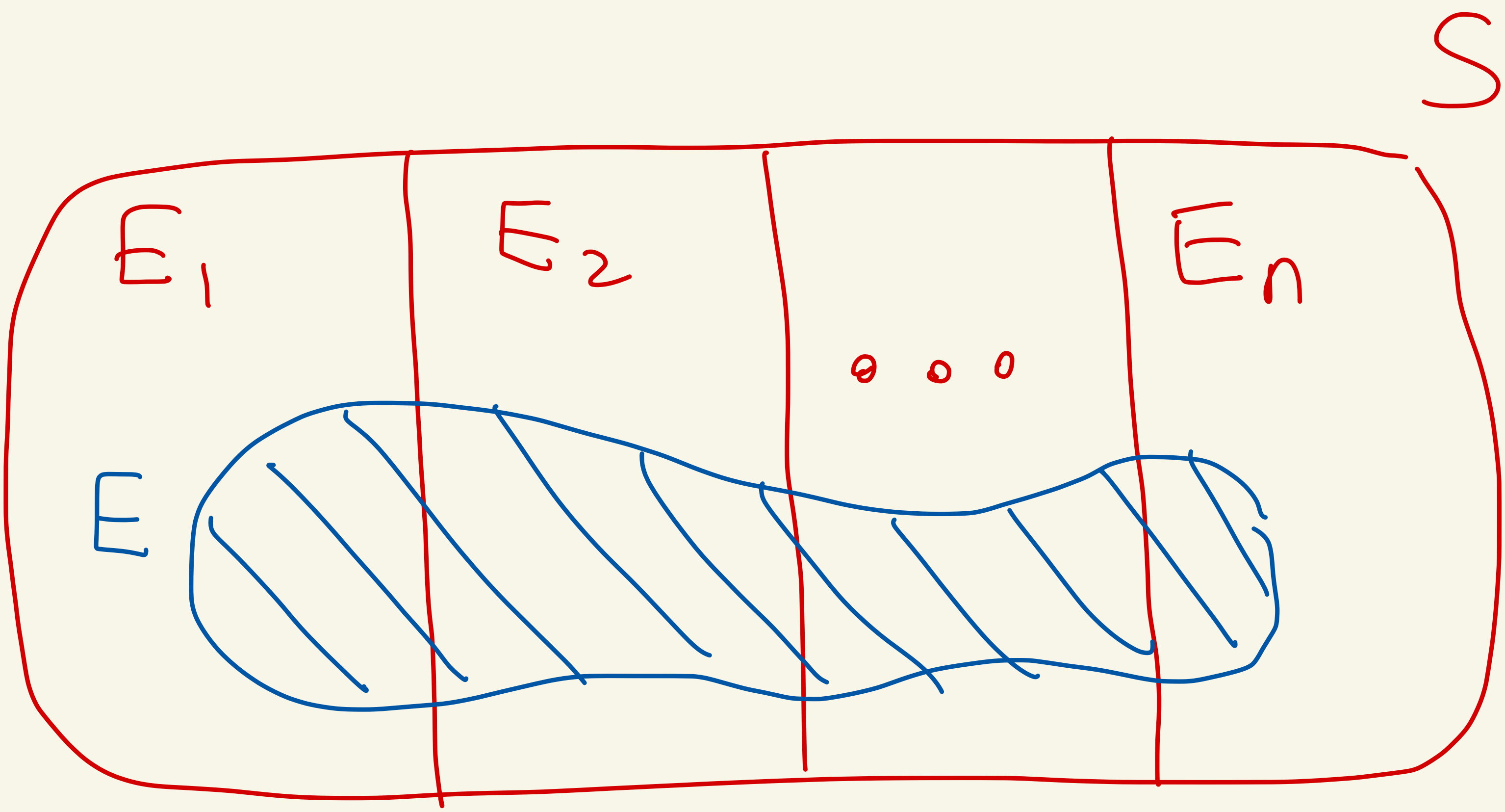
③ (law of total probability)

Suppose $S = E_1 \cup E_2 \cup \dots \cup E_n$ where each $E_i \neq \emptyset$ and $E_i \cap E_j = \emptyset$ if $i \neq j$, and $P(E_i) \neq 0$ for each i

S is broken up into n disjoint events

Then for any event E we have:

$$\begin{aligned} P(E) = {}& P(E|E_1)P(E_1) && \leftarrow P(E\cap E_1)\\ &+ P(E|E_2)P(E_2) && \leftarrow P(E\cap E_2)\\ &+ \cdots && \vdots \\ &+ P(E|E_n)P(E_n) && \leftarrow P(E\cap E_n)\end{aligned}$$



Ex: Suppose there are 3 boxes.
In box 1 are two 4-sided dice.
In box 2 are two 6-sided dice
In box 3 are two 8-sided dice.

Suppose you randomly pick a box (each box is equally likely), then take out the dice from that box and roll them.
What's the probability that the sum of the dice is 8?

Law of total probability

$$P\left(\begin{matrix}\text{sum of}\\ \text{dice is 8}\end{matrix}\right) =$$

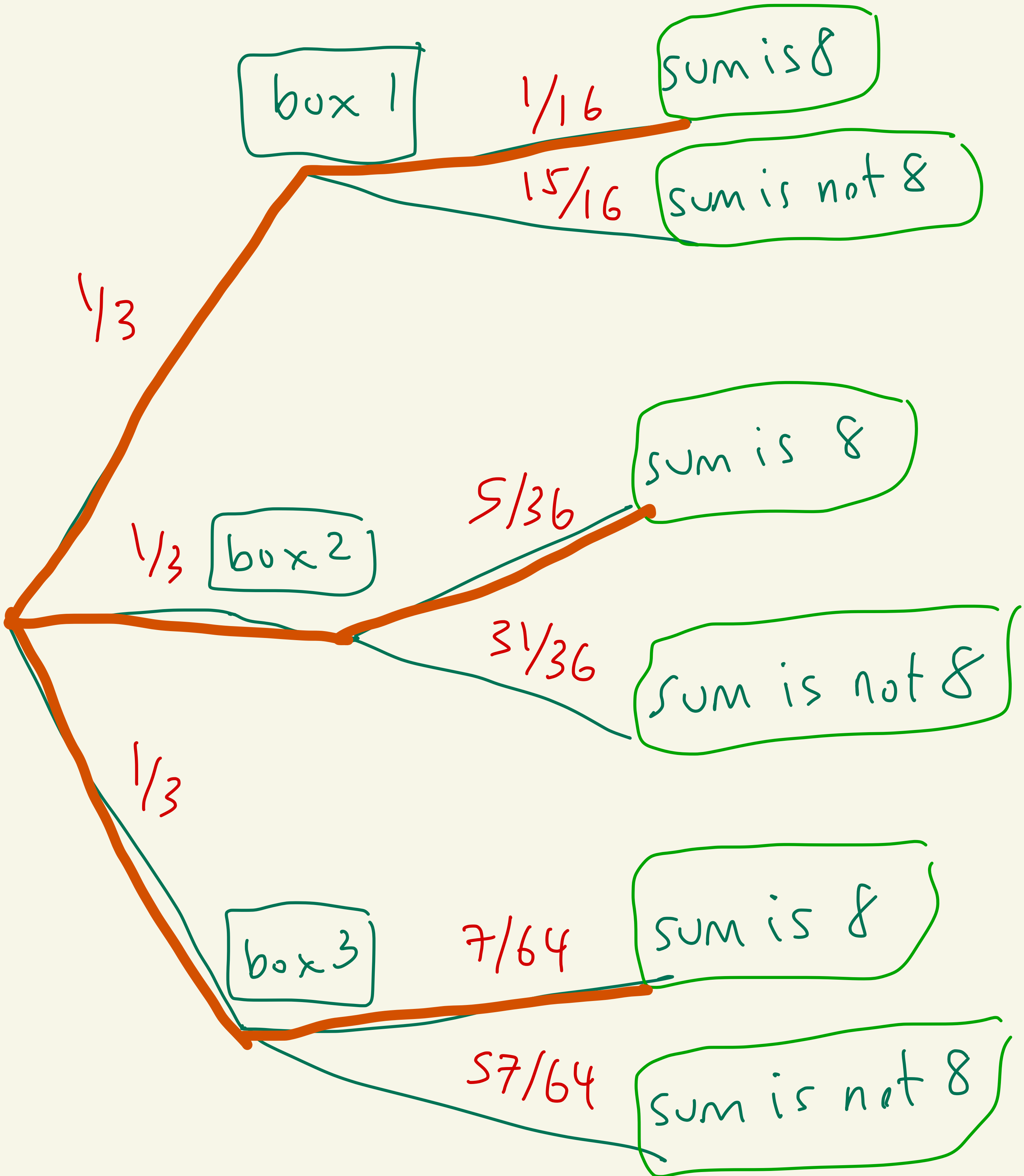
$$= P\left(\begin{matrix}\text{sum of}\\ \text{dice is 8}\end{matrix} \middle| \begin{matrix}\text{box 1}\\ \text{is picked}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{box 1}\\ \text{is}\\ \text{picked}\end{matrix}\right)$$

$$+ P\left(\begin{matrix}\text{sum of}\\ \text{dice is 8}\end{matrix} \middle| \begin{matrix}\text{box 2}\\ \text{is picked}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{box 2}\\ \text{is}\\ \text{picked}\end{matrix}\right)$$

$$+ P\left(\begin{matrix}\text{sum of}\\ \text{dice is 8}\end{matrix} \middle| \begin{matrix}\text{box 3}\\ \text{is picked}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{box 3}\\ \text{is picked}\end{matrix}\right)$$

$$= \underbrace{\left(\frac{1}{16}\right)}_{\substack{\text{4-sided dice}\\ (4,4)}}\left(\frac{1}{3}\right) + \underbrace{\left(\frac{5}{36}\right)}_{\substack{\text{6-sided dice}\\ (2,6),\,(6,2)\\ (3,5),\,(5,3)\\ (4,4)}}\left(\frac{1}{3}\right) + \underbrace{\left(\frac{7}{64}\right)}_{\substack{\text{8-sided dice}\\ (1,7),\,(7,1)\\ (2,6),\,(6,2)\\ (3,5),\,(5,3)\\ (4,4)}}\left(\frac{1}{3}\right)$$

$$= \frac{11{,}456}{110{,}592} \approx \boxed{0.1036 \approx 10.36\%}$$



same answer:

$$\left(\frac{1}{3}\right)\left(\frac{1}{16}\right)+\left(\frac{1}{3}\right)\left(\frac{5}{36}\right)+\left(\frac{1}{3}\right)\left(\frac{7}{64}\right)$$

Ex: (Montey Hall)

Let's redo the probability of winning the montey hall game where we always pick door 1 and switch after Montey reveals a door.

$$
\begin{aligned}
P(\text{win car}) &= P\left(\begin{matrix}\text{win}\\ \text{car}\end{matrix} \,\middle|\, \begin{matrix}\text{car behind}\\ \text{door 1}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{car behind}\\ \text{door 1}\end{matrix}\right) \\
&+ P\left(\begin{matrix}\text{win}\\ \text{car}\end{matrix} \,\middle|\, \begin{matrix}\text{car behind}\\ \text{door 2}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{car behind}\\ \text{door 2}\end{matrix}\right) \\
&+ P\left(\begin{matrix}\text{win}\\ \text{car}\end{matrix} \,\middle|\, \begin{matrix}\text{car behind}\\ \text{door 3}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{car behind}\\ \text{door 3}\end{matrix}\right) \\
&= (0)\left(\tfrac{1}{3}\right) + (1)\left(\tfrac{1}{3}\right) + (1)\left(\tfrac{1}{3}\right) \\
&= 2/3
\end{aligned}
$$

Independent events

Given two events E and F sometimes we get

$$P(E|F) = P(E)$$

this equation is saying: if F occurs then it doesn't change the probability that E will occur

and sometimes we don't get this.

When will this happen?

when

$$\frac{P(E \cap F)}{P(F)} = P(E)$$

or when

$$\boxed{P(E \cap F) = P(E)P(F)}$$

Def: We say that the events E and F are **independent** if

$$P(E \cap F) = P(E)\,P(F)$$

Ex: Suppose you roll two 4-sided die. Let E be the event that die 1 lands on 4. Let F be the event that die 2 lands on 4. Are these events independent?

$E = \{(4,1),(4,2),(4,3),(4,4)\}$

$F = \{(1,4),(2,4),(3,4),(4,4)\}$

$E \cap F = \{(4,4)\}$

$P(E \cap F) = 1/16$

$$P(E)P(F) = \left(\frac{4}{16}\right)\left(\frac{4}{16}\right) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$$

EQUAL

Yes, the events are independent

<u>Note:</u> Suppose $P(E) > 0$, $P(F) > 0$.

Then,

$\boxed{\text{E and F are independent}}$

is equivalent to

$$\boxed{P(E \cap F) = P(E)P(F)}$$

is equivalent to

$$\boxed{\frac{P(E\cap F)}{P(E)} = P(F) \quad \text{and} \quad \frac{P(E\cap F)}{P(F)} = P(E)}$$

is equivalent to

$$\boxed{P(F|E) = P(F) \quad \text{and} \quad P(E|F) = P(E)}$$