Math 4740
2/5/25

Topic 2 - Counting and Probability

Basic counting principle

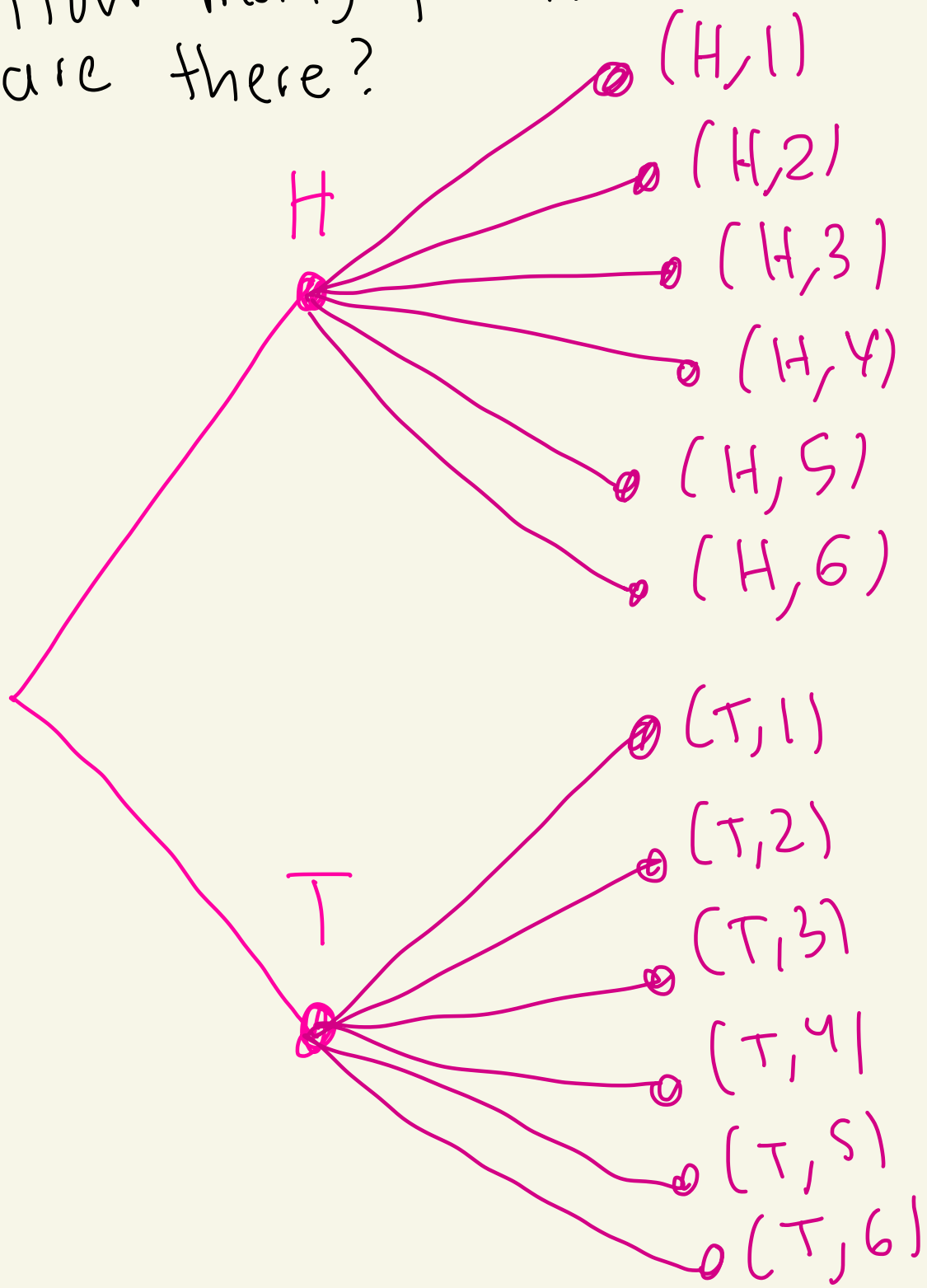
If r experiments are performed in a row such that the first experiment may result in n_1 possible outcomes; and if for each of these n_1 possible outcomes there are n_2 possible outcomes for the second experiment; and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes for the third experiment; and if $\dots$,

then there are

$$n_1 \cdot n_2 \cdot n_3 \cdots n_r$$

possible outcomes for the r experiments.

Ex: Suppose we flip a coin and then roll a 6-sided die. How many possible outcomes are there?



total # of outcomes is $n_1 \cdot n_2 = 12$

$n_1 = 2$

$n_2 = 6$

Can write it

H/T 1/2/3/4/5/6

$2 \cdot 6 = 12$

Ex: In CA, a license plate consists of one number (0-9) followed by three upper-case letters, followed by three numbers. The only exclusion is that the letters I, O, and Q cannot be used in spot 2

or spot 4.

Examples are:

4 A A B 1 4 1
1 2 3 4 5 6 7

3 X Z A 2 2 2

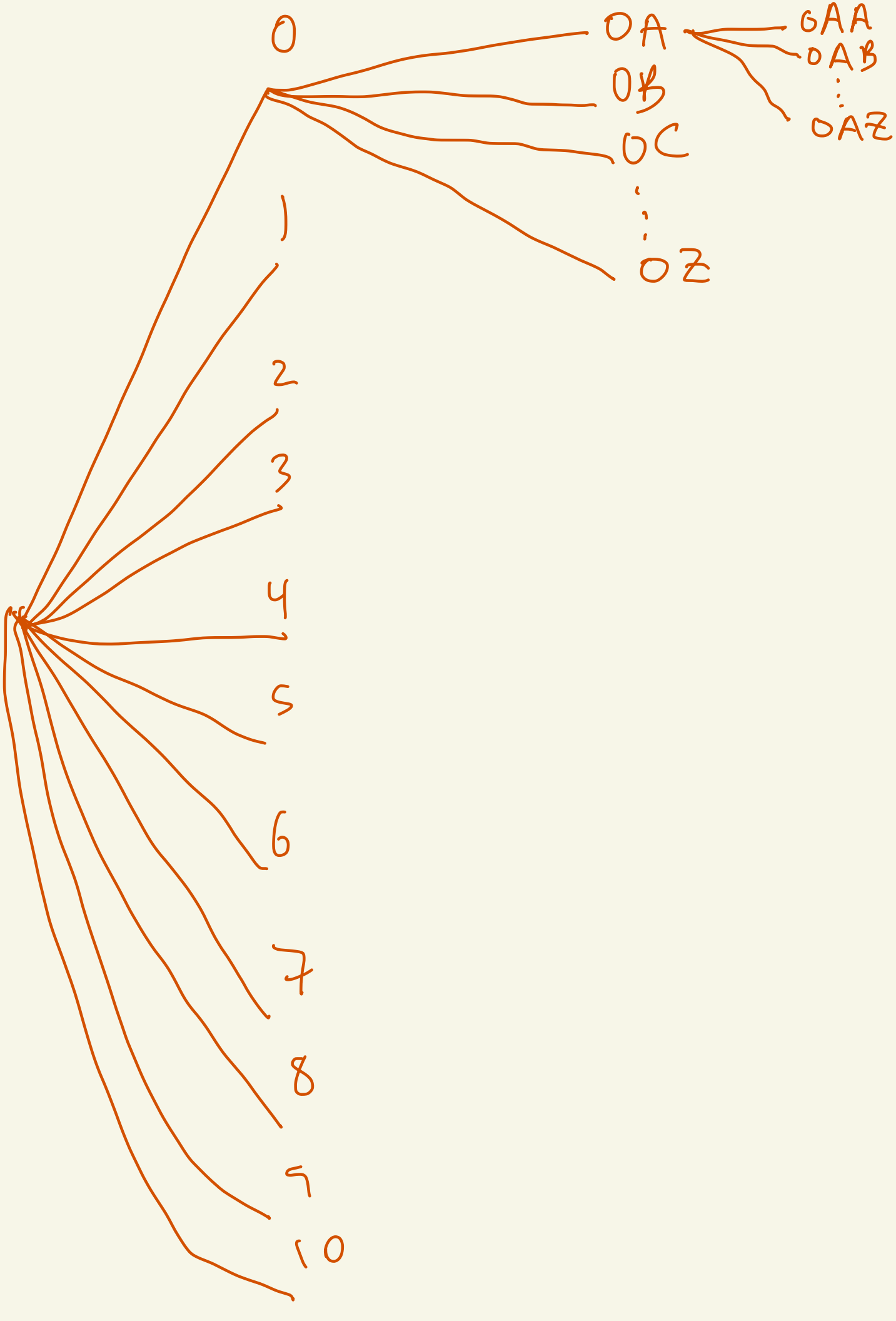
How many license plates are there?

$$\underset{1}{10} \cdot \underset{2}{23} \cdot \underset{3}{26} \cdot \underset{4}{23} \cdot \underset{5}{10} \cdot \underset{6}{10} \cdot \underset{7}{10}$$

not I, O, Q

Answer $= 23^2 \cdot 26 \cdot 10^4$

$= 137,540,000$


0
OA
OAA
OAB
OAZ
OB
OC
OZ
1
2
3
4
5
6
7
8
9
10

Birthday Paradox

Suppose there are N people in a room. What is the probability that at least two people have the same birthday?

> We mean same month/day year is not included

Assumptions:

1. We will assume that no one has Feb 29 as their bday this a leap year.
2. We will assume each day is equally likely.
3. Assume $N \leq 365$ since if $N > 365$ then the answer is 100%.

Sample space for N=3

$$S = \{ (\underbrace{\text{date}\ 1}_{\text{person 1}}, \underbrace{\text{date}\ 2}_{\text{person 2}}, \underbrace{\text{date}\ 3}_{\text{person 3}}) \}$$

$$= \{ (\text{Feb}\ 1, \text{April}\ 9, \text{Jan}\ 3), \ldots \}$$

(no one has same bday)

$$(\text{Mar}\ 1, \text{May}\ 3, \text{Mar}\ 1), \ldots \}$$

(two have same bday)

$$|S| = (365)^3 \leftarrow \boxed{N=3}$$

In general for N people we have $|S| = (365)^N$.

Let E be the event that at least two people have the same birthday.

Since all days are equally likely the probability that E will occur is $\frac{|E|}{|S|}$.

Too hard to count.

Instead we will calculate the probability of $\overline{E}$

where $\overline{E}$ is when no two people have the same birthday.

We will use:

$$\frac{|E|}{|S|} = 1 - \frac{|\overline{E}|}{|S|}$$

$\frac{|E|}{|S|}$: probability of E

$1 - \frac{|\overline{E}|}{|S|}$: probability of $\overline{E}$

Let's count $|\overline{E}|$.

$$\frac{365}{\text{person 1}} \cdot \frac{364}{\text{person 2}} \cdot \frac{363}{\text{person 3}} \cdots \frac{(365-N+1)}{\text{person } N}$$

- person 2: can't have person 1's bday
- person 3: can't have person 1 or person 2 bday
- person N: can't have any previous bday

So,

$$|E| = (365)(364)(363)\cdots(365-N+1)$$

Thus,

$$P(E) = 1 - \frac{|E|}{|S|}$$

$$= 1 - \frac{(365)(364)(363)\cdots(365-N+1)}{(365)^N}$$

$$= 1 - \frac{365!}{(365)^N (365-N)!}$$

Ex for $N=3$:

$$P(E) = 1 - \frac{365!}{(365)^3 (362)!}$$

$$= 1 - \frac{365 \cdot 364 \cdot 363}{(365)^3}$$

SEE WEBSITE FOR TABLE

Permutations

Suppose you have n objects.
A permutation of the n objects is an ordered list of the n objects

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Ex: What are all the permutations of a, b, c?

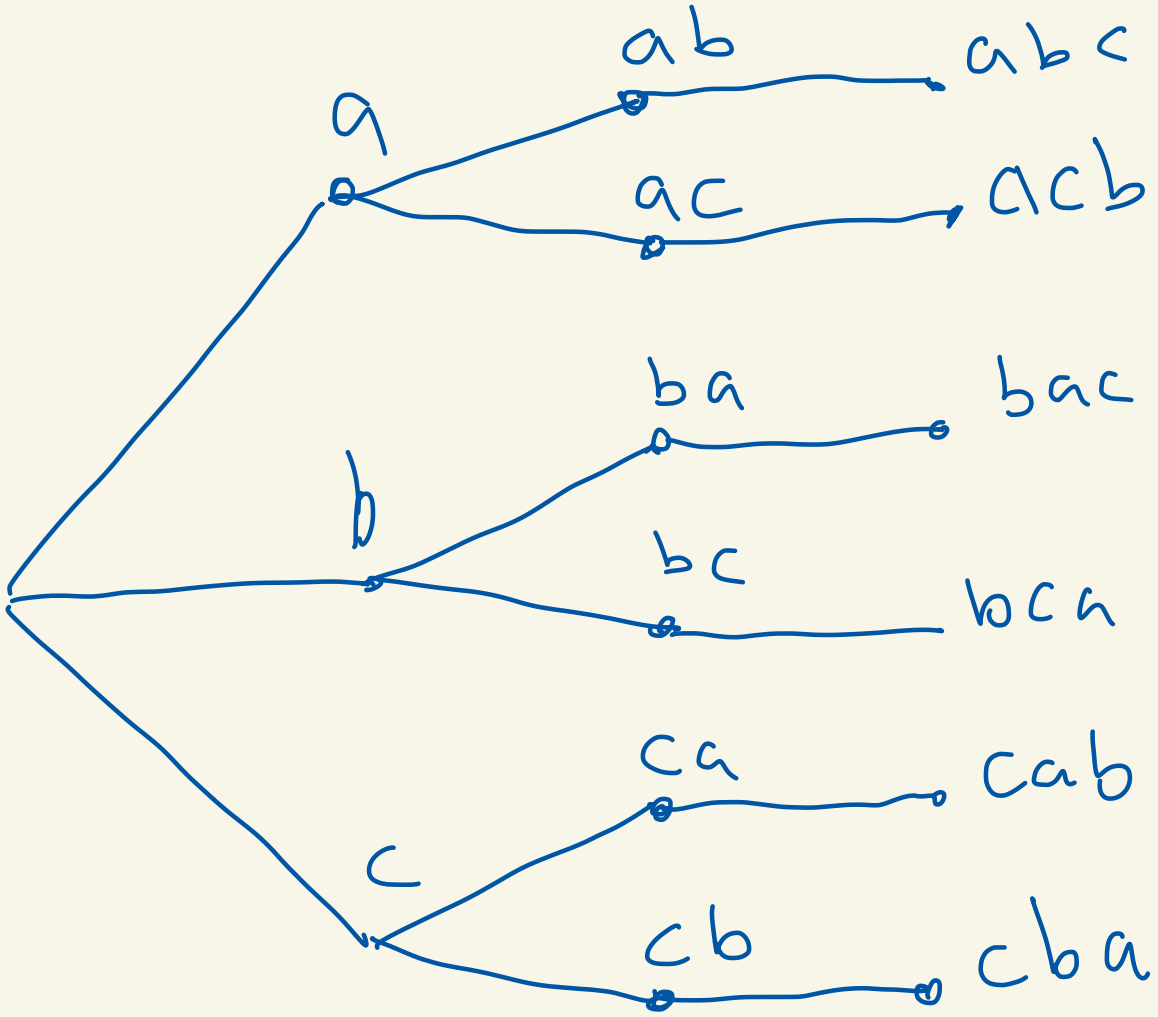
permutations	math way
abc	(a, b, c)
acb	(a, c, b)
bac	(b, a, c)
bca	(b, c, a)
cab	(c, a, b)
cba	(c, b, a)

How many are there? 6

$$3 \cdot 2 \cdot 1 = 3!$$

↑ can't be same as spot 1

↑ can't be same as previous



In general, for n objects there are n! permutations