4740

3/12/24

(topic 4 continued...)

We will talk about the St. Petersburg Paradox. Goes back to the 1700's

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A casino offers a game to a single player. A coin is tossed at each stage. The pot (amount won) starts at \$2 and doubles every time a head is flipped. The first time a tail is flipped the game ends and the player wins whats in the pot. How much would you pay to play this game? You don't get back what you pay, just the money you win from the pot.

Ex: pay $5 to play

pot	flip
$2	H
$4	H
$8	H
$16	H
$32	T

pot $32
bet $5

win = $27

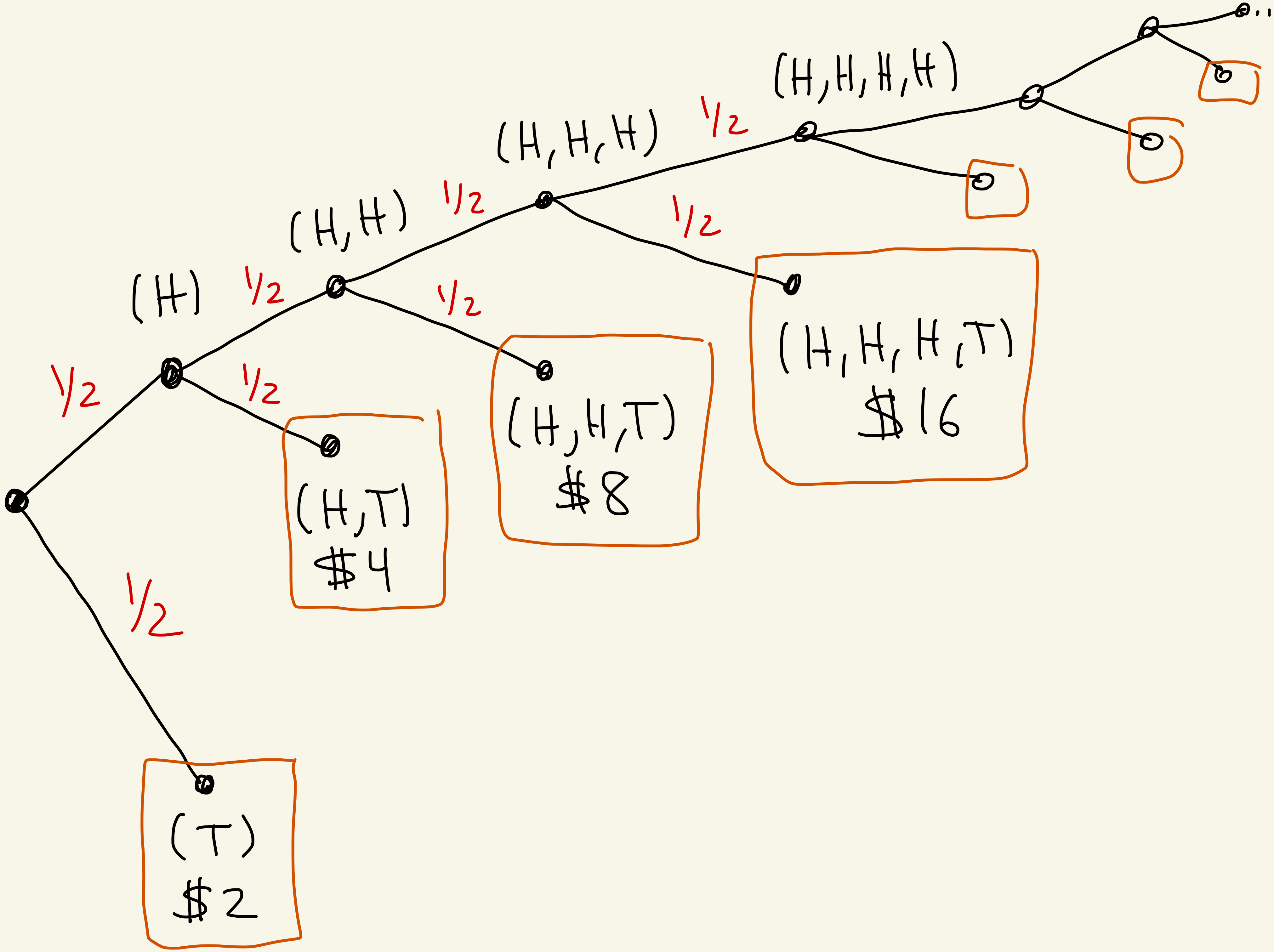
Ex: pay $5

pot	flip
$2	T

pot $2
bet $5

lost $3

Let's calculate the expected value of this game.



(H)
(H,H)
(H,H,H)
(H,H,H,H)
1/2
(T)
$2
(H,T)
$4
(H,H,T)
$8
(H,H,H,T)
$16

Let X be the amount won/lost on this game.

$$E[X] = (-\$\text{amount paid to play})\underbrace{(1)}_{\substack{100\% \\ \text{of time}}}$$

$$+ (\$2)\underbrace{\left(\frac{1}{2}\right)}_{\substack{\text{probability} \\ \text{of winning} \\ \$2}} + (\$4)\underbrace{\left(\frac{1}{4}\right)}_{\substack{\text{probability} \\ \text{of} \\ \text{winning } \$4}}$$

$$+ (\$8)\left(\frac{1}{8}\right) + (\$16)\left(\frac{1}{16}\right)$$

$$+ \dots$$

$$= (-\$\text{amount paid to play})$$

$$+ \underbrace{\$1 + \$1 + \$1 + \$1 + \dots}_{\text{infinite \# of } \$1\text{'s}}$$

$$= \infty$$

So we get infinite expected value.
But it's hard to win alot.
For example suppose we want the pot to be at least
$\$2^{20} = \$1{,}048{,}576$
The probability this would happen is

$$\underbrace{\frac{1}{2^{20}}}_{\substack{\text{prob.}\\ \text{win}\\ \$2^{20}}} + \underbrace{\frac{1}{2^{21}}}_{\substack{\text{prob.}\\ \text{win}\\ \$2^{21}}} + \underbrace{\frac{1}{2^{22}}}_{\substack{\text{prob}\\ \text{win}\\ \$2^{22}}} + \cdots$$

$$= \frac{1}{2^{20}}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2^{20}}\left[\frac{1}{1-1/2}\right] = \frac{1}{2^{19}} \approx 0.000001907\ldots$$

Geometric sum

$$\boxed{1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} \quad \text{if } -1 < r < 1}$$

Topic 5 - Binomial Random Variables

A Bernoulli trial is an experiment with two outcomes: success or failure.

Suppose success occurs with probability p. Then failure occurs with probability $1-p$.

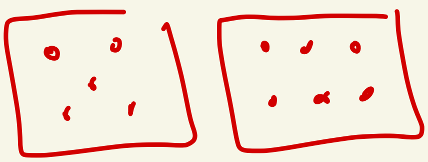
<u>Ex:</u> Experiment is flipping a coin.

success = tails, $p = 1/2$

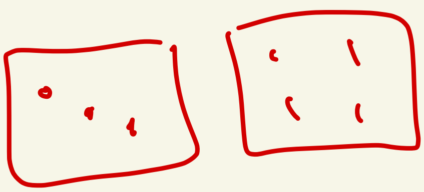
failure = heads, $1-p = 1/2$

<u>Ex:</u> Rolling two 6-sided dice.
Let success be that the sum of the dice is 7 or 11.
And failure is any other sum.

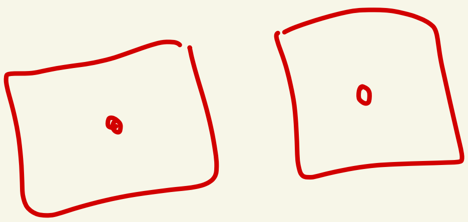
<u>dice roll</u>



← sum is 11, success



← sum is 7, success



← sum is 2, failure

probability of success

$$p = \underbrace{\frac{2}{36}}_{\text{prob. of } 11} + \underbrace{\frac{6}{36}}_{\text{prob. of } 7} = \frac{8}{36}$$

probability of failure

$$1-p = 1-\frac{8}{36} = \frac{28}{36}$$

Ex: Experiment is playing one round of Roulette with the American wheel.

Let success be that the ball lands on a black number.

Then failure is that the ball lands on a red or green number.

Success

$$p = \frac{18}{38} = \frac{9}{19}$$

$$1-p = \frac{20}{38} = \frac{10}{19}$$

Failure

Roulette

2 greens
18 red
18 black

38 total