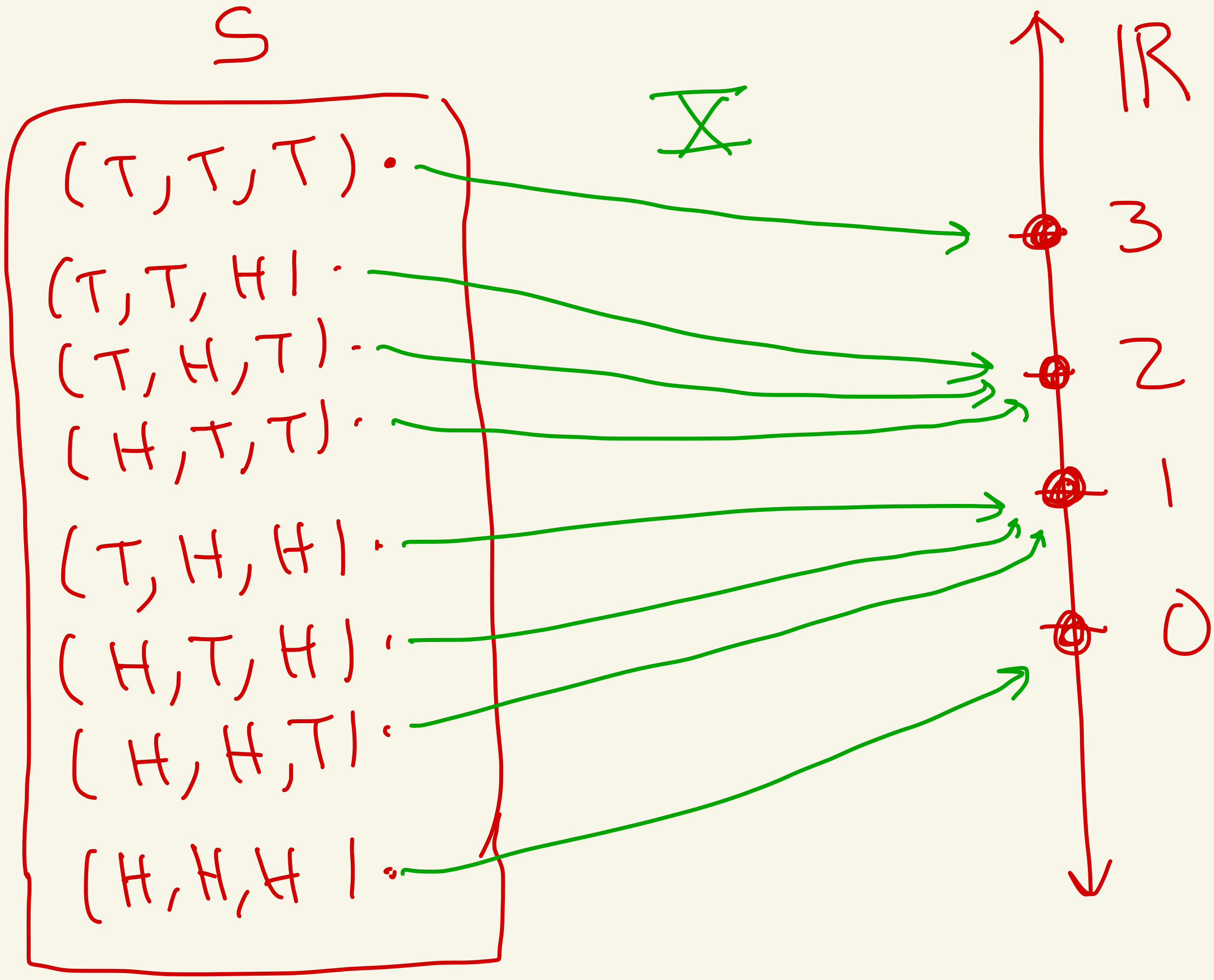
Math 4740

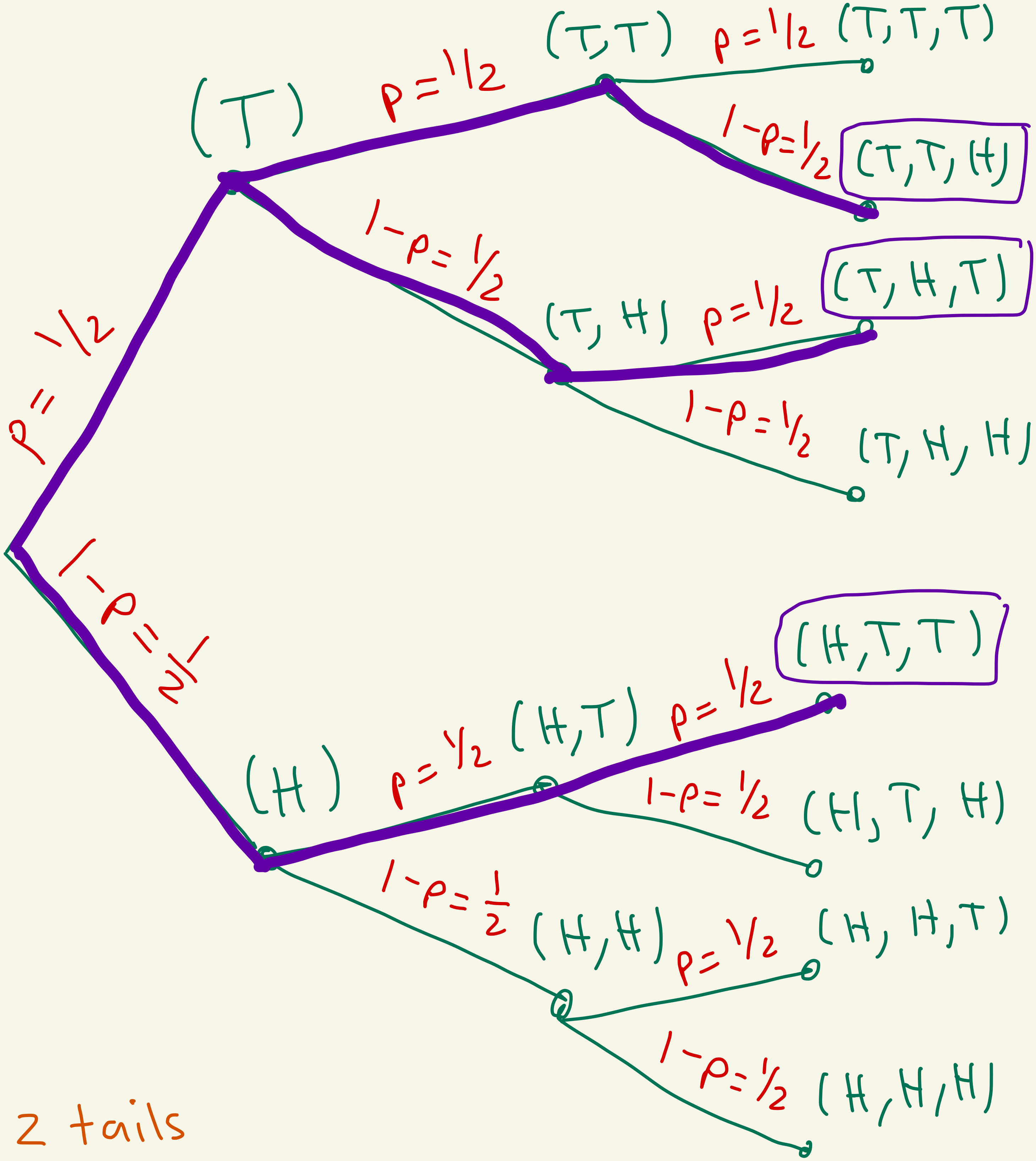
3/24/25

(Topic 5 continued...)

Now suppose that n Bernoulli trials, each with success probability p, are performed in a row independently of each other. Let X be the number of successes. Then X is called a <u>binomial random variable</u> with parameters n and p.

Ex: Suppose you flip a coin $n=3$ times. Where on each flip success is T (tails) with probability $p = 1/2$. Let X be the total number of successes/tails. Then X is a binomial random variable with parameters $n=3$, $p=1/2$.





$$P(\overbrace{X = 2}^{\text{2 tails}}) = p \cdot p \cdot (1-p) + p(1-p)p$$

$$+ (1-p) \cdot p \cdot p$$

$$= 3p^2(1-p)^1$$

$$= \underbrace{\binom{3}{2}}_{\text{picking 2 of 3 spots for the tails}} p^2 (1-p)^{3-2}$$

successes

failures or n-success

<u>Theorem:</u> Let X be a binomial random variable with parameters n and p. Then

$$\underbrace{P(X=k)}_{\text{probability of } k \text{ successes in } n \text{ trials}} = \binom{n}{k} p^k (1-p)^{n-k}$$

when $0 \leq k \leq n$

$\boxed{\text{proof in notes online if interested}}$

<u>Ex:</u> Suppose we flip a coin 100 times. What is the probability of exactly 48 tails occuring?

$n = 100 \leftarrow$ # flips

$k = 48 \leftarrow$ # successes / tails

$p = \frac{1}{2} \leftarrow$ prob. of success / tail

$1 - p = \frac{1}{2} \leftarrow$ prob. of failure / heads

X = # tails in $n = 100$ flips

$$P(X = 48) = \binom{100}{48} \cdot \left(\tfrac{1}{2}\right)^{48} \cdot \left(1 - \tfrac{1}{2}\right)^{100-48}$$

$$= \binom{100}{48} \cdot \left(\frac{1}{2^{48}}\right)\left(\frac{1}{2^{52}}\right)$$

$$= \binom{100}{48} \cdot \frac{1}{2^{100}}$$

$$= \frac{93,206,558,875,049,876,949,581,681,100}{1,267,650,600,228,229,401,496,703,205,376}$$

$$\approx 0.0735\ldots \approx \boxed{7.35\%}$$

Ex: Suppose we flip a coin 20 times. What is the probability of getting between 10 and 12 tails?

$n = 20$

$p = 1/2 \leftarrow$ prob. of tail/success

$1-p = 1/2 \leftarrow$ prob. of failure/heads

$X = \#$ of tails/successes in $n=20$ flips

$$P(10 \le X \le 12) =$$

$$= P(X=10) + P(X=11) + P(X=12)$$

$$= \underbrace{\binom{20}{10}\cdot\left(\frac{1}{2}\right)^{10}\cdot\left(\frac{1}{2}\right)^{10}}_{\binom{20}{10}\cdot\left(\frac{1}{2}\right)^{10}\cdot\left(1-\frac{1}{2}\right)^{20-10}} + \binom{20}{11}\cdot\left(\frac{1}{2}\right)^{11}\left(\frac{1}{2}\right)^{9}$$

$$+ \binom{20}{12}\cdot\left(\frac{1}{2}\right)^{12}\cdot\left(\frac{1}{2}\right)^{8}$$

$$= \frac{\binom{20}{10} + \binom{20}{11} + \binom{20}{12}}{2^{20}}$$

$$= \frac{184{,}756 + 167{,}960 + 125{,}970}{1{,}048{,}576}$$

$$\approx \boxed{0.456511\ldots} \approx \boxed{45.6511\%}$$

Ex: Suppose we play Roulette. We bet 5 times in a row and each time we bet on red. Let X be the number of times we win (# times red occurs). Calculate the probability that we win at least 3 times.

$n = 5$

$p = \frac{18}{38}$ ← prob. of red/success each spin of wheel

red = 18
black = 18
green = 2

X = # reds/successes that occur

$1 - p = \frac{20}{38}$ ← prob. of failure green/black

$$P(X \geq 3)$$

$$= P(X=3) + P(X=4) + P(X=5)$$

$$= \binom{5}{3} \cdot \left(\frac{18}{38}\right)^3 \cdot \left(\frac{20}{38}\right)^2$$

$$+ \binom{5}{4} \cdot \left(\frac{18}{38}\right)^4 \cdot \left(\frac{20}{38}\right)^1$$

$$+ \binom{5}{5} \cdot \left(\frac{18}{38}\right)^5 \cdot \left(\frac{20}{38}\right)^0$$

$$= \frac{10 \cdot 18^3 \cdot 20^2 + 5 \cdot 18^4 + 20^1 + 1 \cdot 18^5 \cdot 20^0}{38^5}$$

$$= \frac{35{,}715{,}168}{79{,}235{,}168}$$

$$\approx \boxed{0.45} \approx \boxed{45\%}$$

<u>Theorem:</u> Let X be a binomial random variable with parameters n and p.

Then, $E[X] = np$

proof in notes online

<u>Ex:</u> Suppose you flip a coin $n = 100$ times and success is tails with probability $p = 1/2$.

X = total # of tails.

$$E[X] = np = (100)\left(\tfrac{1}{2}\right) = 50$$