4740
3/5/25

Topic 4 – Random Variables and Expected Value

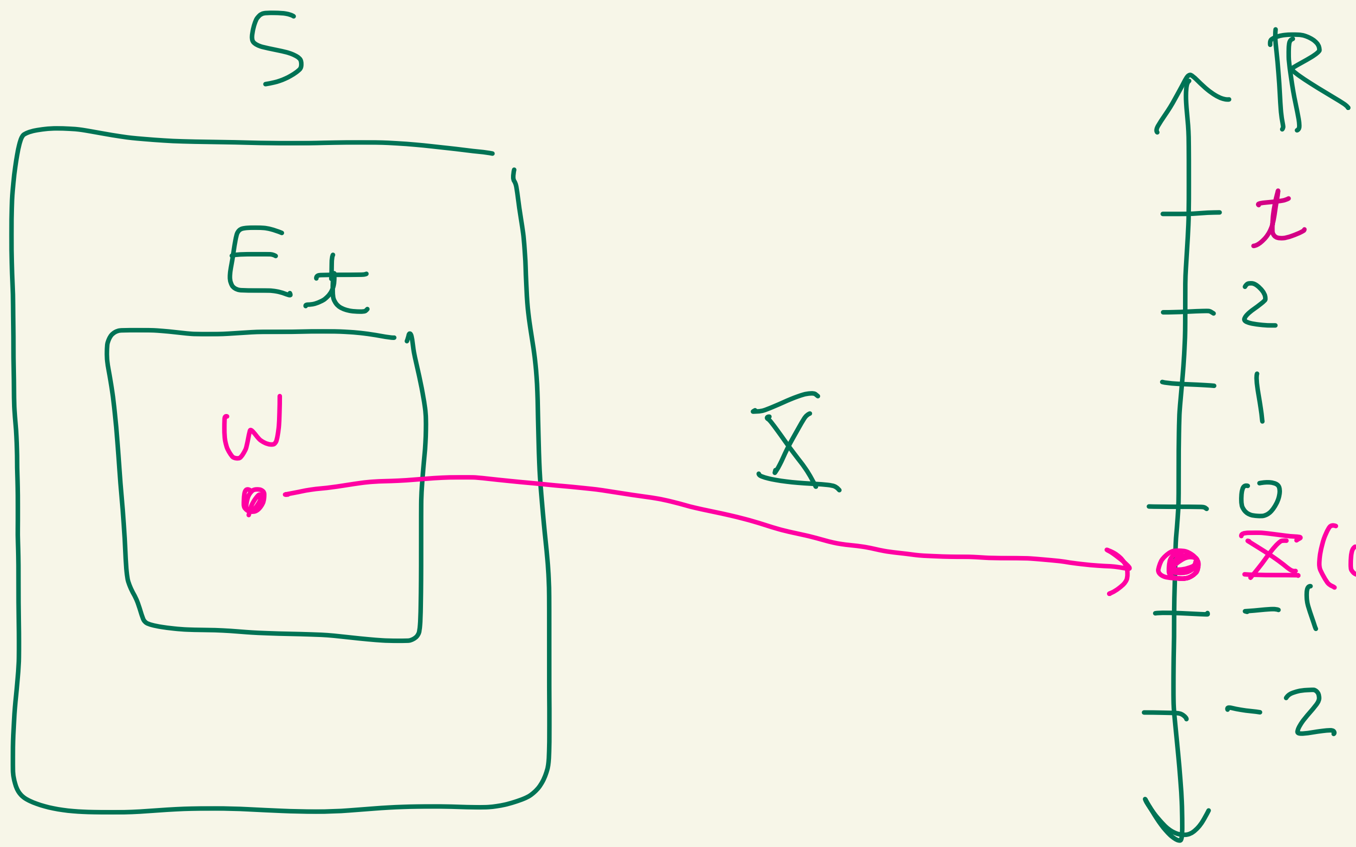
Def: Let (S, Ω, P) be a probability space. A random variable is a function $\underbrace{X : S \longrightarrow \mathbb{R}}$

means: S is the input to X, and X outputs real numbers

such that for all real numbers t we have that

$$E_t = \{ w \mid w \in S \text{ and } X(w) \leq t \}$$

is an event in Ω.



Note: The condition on E_t means we can calculate $P(E_t)$. In our class when S is finite and Ω is all subsets of S this condition will always be satisfied. So for us, a random variable is just a function

$$X : S \longrightarrow \mathbb{R}$$

Def: Let $X : S \to \mathbb{R}$ be a random variable. We say that X is discrete if the range of X can be enumerated as a list of values:

$$x_1, x_2, x_3, x_4, \ldots$$

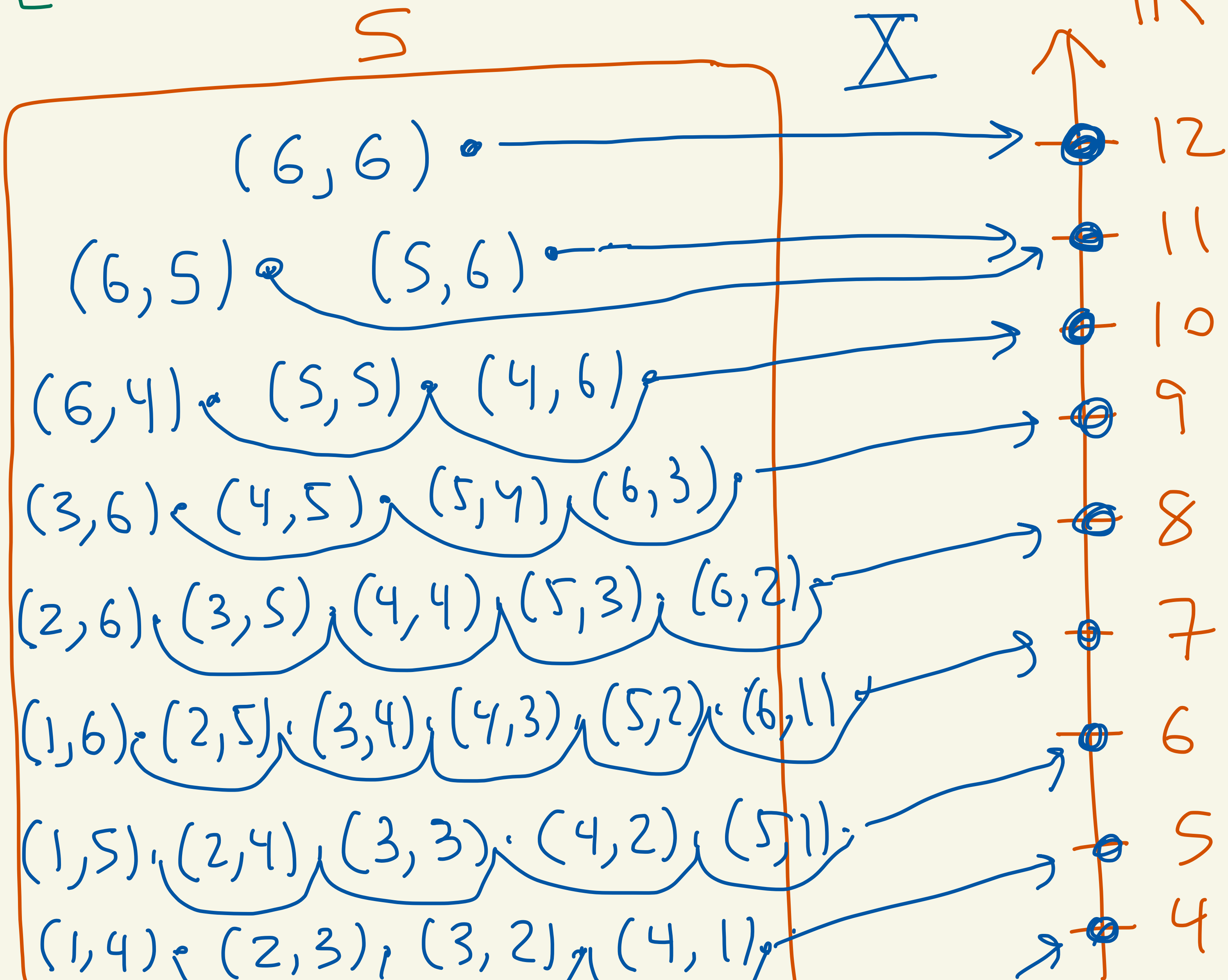
(it can be an infinite or finite list)

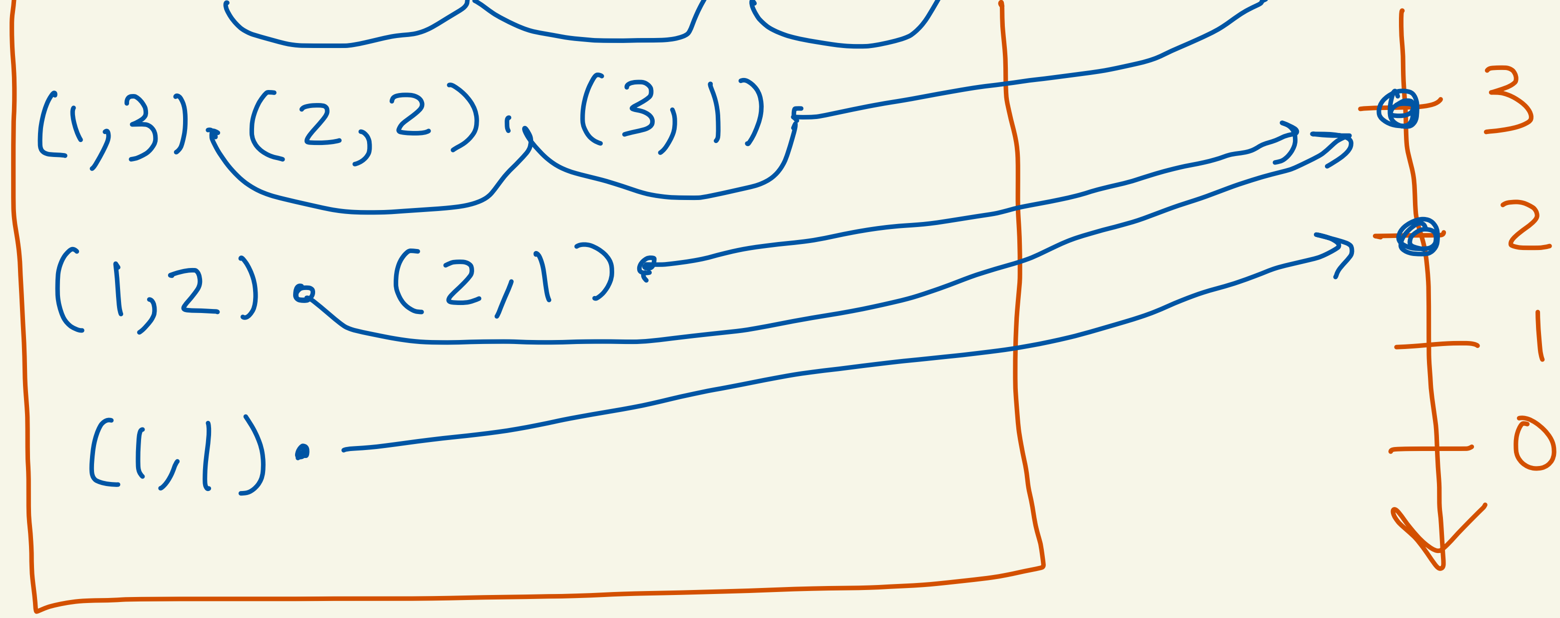
[For 3450 people: discrete means the range of X is countable]

Ex: Let (S, Ω, P) be the probability space for rolling two 6-sided dice.

Let $X : S \to \mathbb{R}$ be the sum of the dice.

[Ex: $X(4,2) = 4 + 2 = 6$]





X is discrete because its range is 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

Def: Let (S, Ω, P) be a probability space. Let X be a random variable on the space.

Define:

$$P(X = i) = \underbrace{P(\{\omega \mid \omega \in S \text{ and } X(\omega) = i\})}$$

take the probability of all ω that get send to i under X

$$P(X \leq i) = P(\{\omega \mid \omega \in S \text{ and } X(\omega) \leq i\})$$

Similarly you can define $P(X < i)$, $P(X \geq i)$, etc.

The <u>probability function</u> p of X is $p(i) = P(X = i)$

<u>Ex:</u> Let (S, Ω, P) be rolling two 6-sided dice. Let X be the sum of the dice.

Let's calcule the probability function $p(i) = P(X = i)$.

We get

$$p(2) = P(X = 2) = P(\{(1,1)\}) = 1/36$$

$$p(3) = P(X=3) = P(\{(1,2),(2,1)\}) = 2/36$$

$$p(4) = P(X=4) = P(\{(1,3),(2,2),(3,1)\}) = 3/36$$

$$p(5) = P(X=5) = P(\{(1,4),(2,3),(3,2),(4,1)\}) = \frac{4}{36}$$

$$p(6) = P(X=6) = P(\{(1,5),(2,4),(3,3),(4,2),(5,1)\}) = 5/36$$

$$p(7) = P(X=7) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = 6/36$$

$$p(8) = P(X=8) = 5/36$$

$$p(9) = P(X=9) = 4/36$$

$$p(10) = P(X=10) = 3/36$$

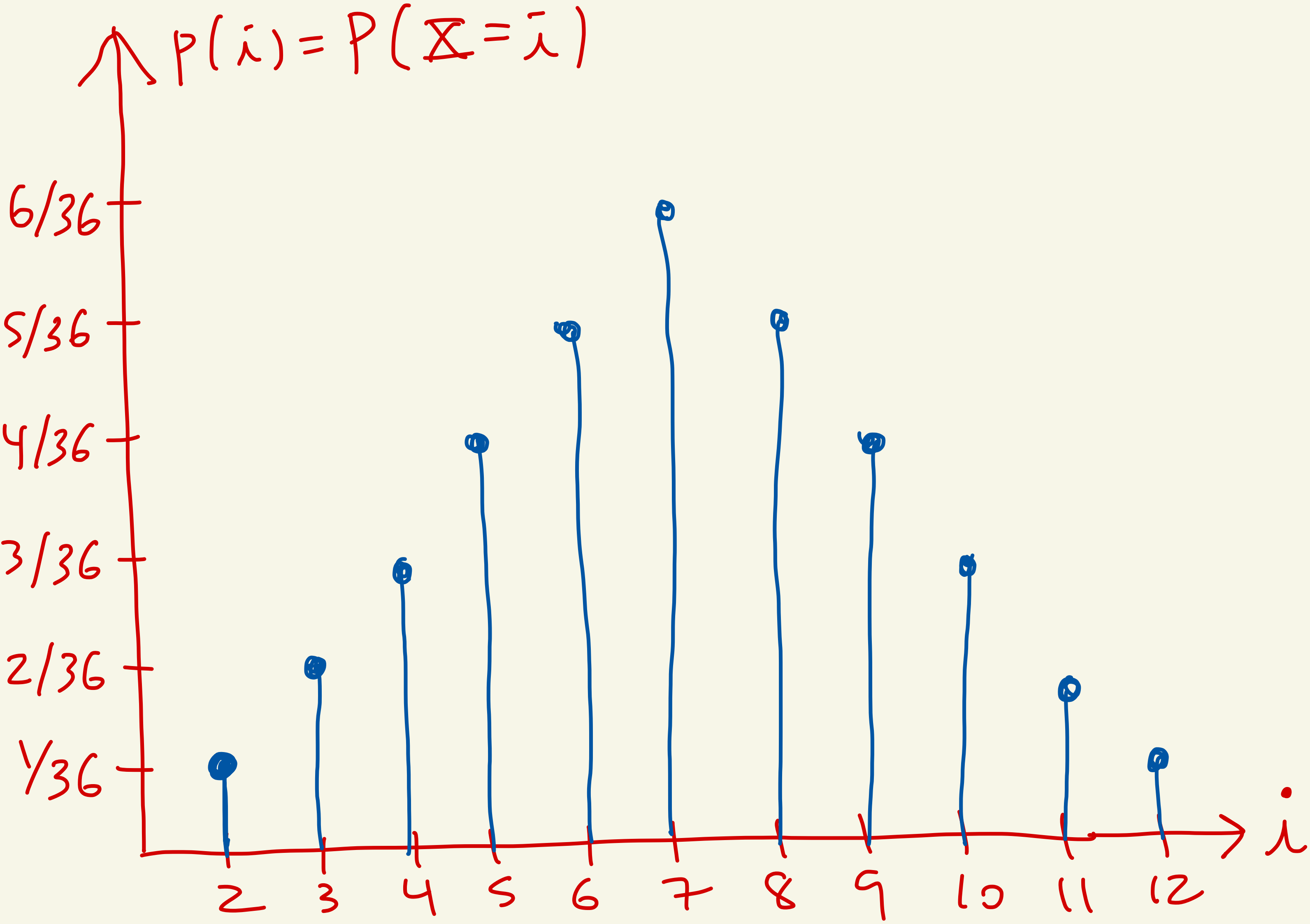
$$p(11) = P(X=11) = 2/36$$

$$p(12) = P(X=12) = 1/36$$

$$P(X \leq 3) = P(\{(1,1),(2,1),(1,2)\}) = \frac{3}{36}$$

$$= P(X=2) + P(X=3)$$

PICTURE TIME!



Def: Let X be a discrete random variable on a probability space (S, Ω, P). Let $x_1, x_2, x_3, \ldots$ be the range of X (outputs of X).
The expected value of X is

$$E[X] = \sum_i x_i \cdot P(X = x_i)$$

Ex: Let (S, Ω, P) represent rolling two 6-sided dice and X be the sum of the dice. Then,

$$E[X] = (2)\underbrace{\left(\frac{1}{36}\right)}_{P(X=2)} + (3)\underbrace{\left(\frac{2}{36}\right)}_{P(X=3)} +$$

$$+ (4)\underbrace{\left(\frac{3}{36}\right)}_{P(X=4)} + (5)\underbrace{\left(\frac{4}{36}\right)}_{P(X=5)}$$

$$+ (6)\underbrace{\left(\frac{5}{36}\right)}_{P(X=6)} + (7)\underbrace{\left(\frac{6}{36}\right)}_{P(X=7)} + (8)\underbrace{\left(\frac{5}{36}\right)}_{P(X=8)}$$

$$+ (9)\underbrace{\left(\frac{4}{36}\right)}_{P(X=9)} + (10)\underbrace{\left(\frac{3}{36}\right)}_{P(X=10)} + (11)\underbrace{\left(\frac{2}{36}\right)}_{P(X=11)}$$

$$+ (12)\underbrace{\left(\frac{1}{36}\right)}_{P(X=12)}$$

$$= \frac{252}{36} = \boxed{7}$$