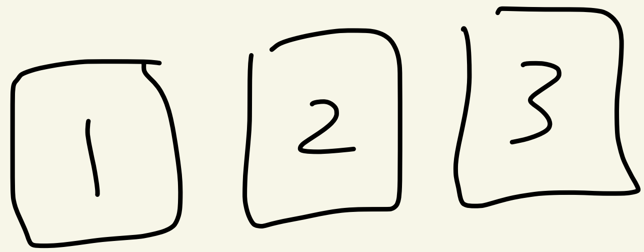
Math 4740

9/25/24

Topic 3 – Conditional Probability

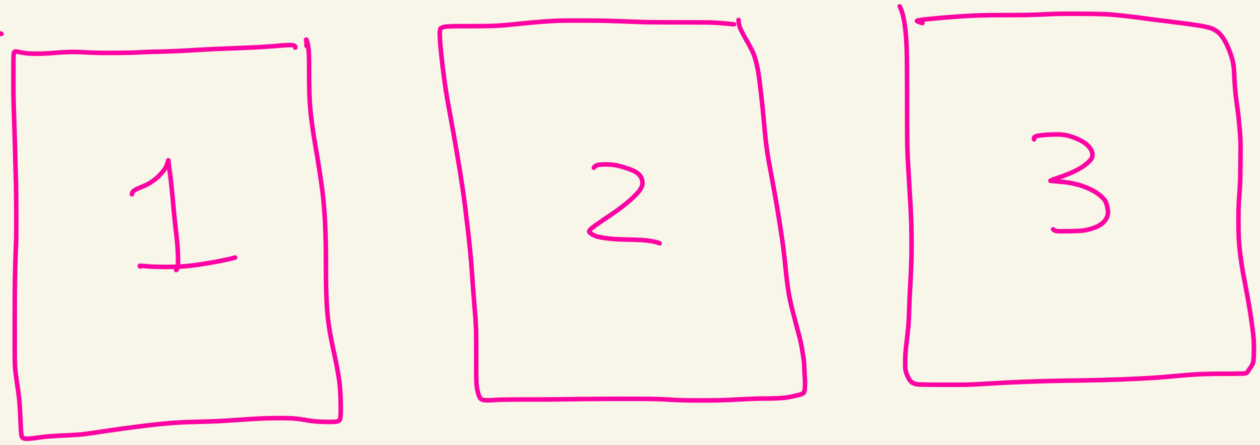
Montey Hall Problem

- Three doors 

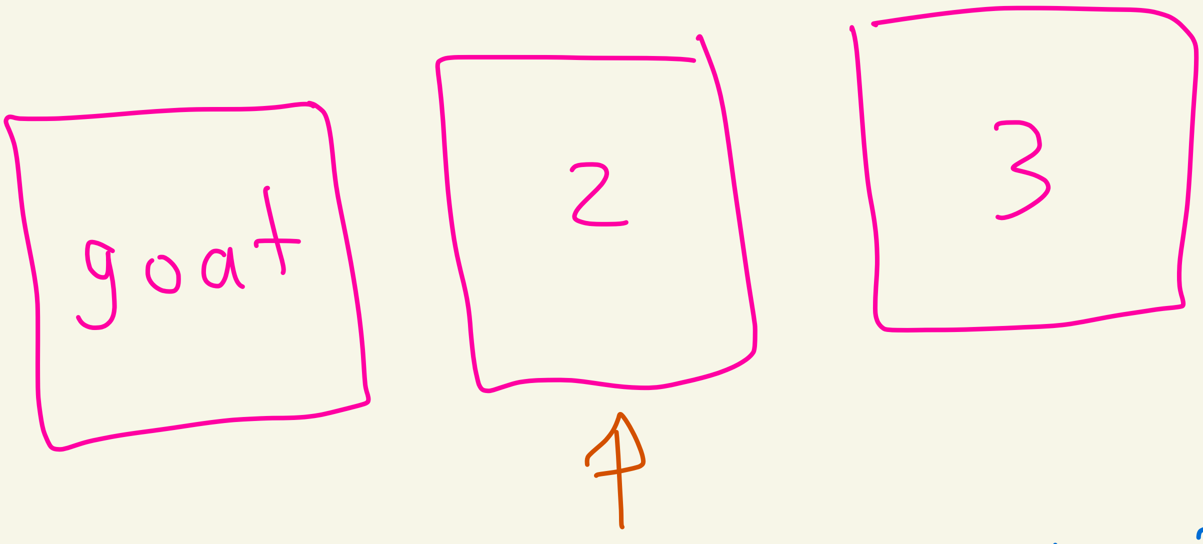
  behind doors: car, goat, goat

- You pick a door.
  MH reveals a door with
  a goat. Then asks:
  do you want to keep
  your door or switch doors?
  You're stuck with the prize
  behind the door you pick.

Ex:

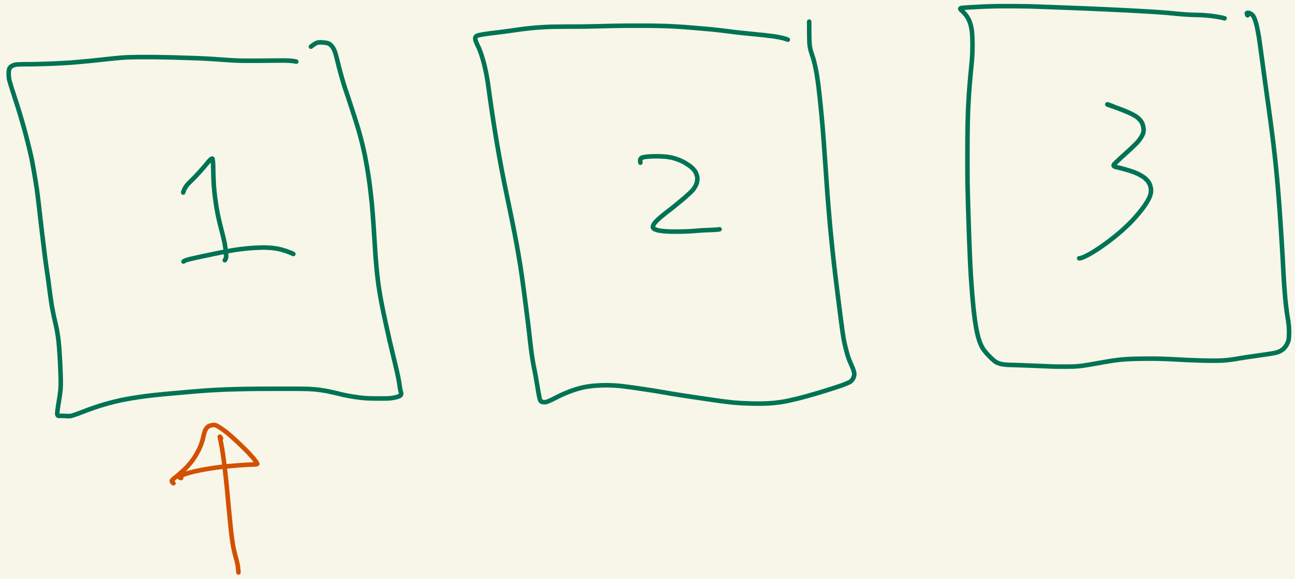


door 2 picked.
door 1 revealed.



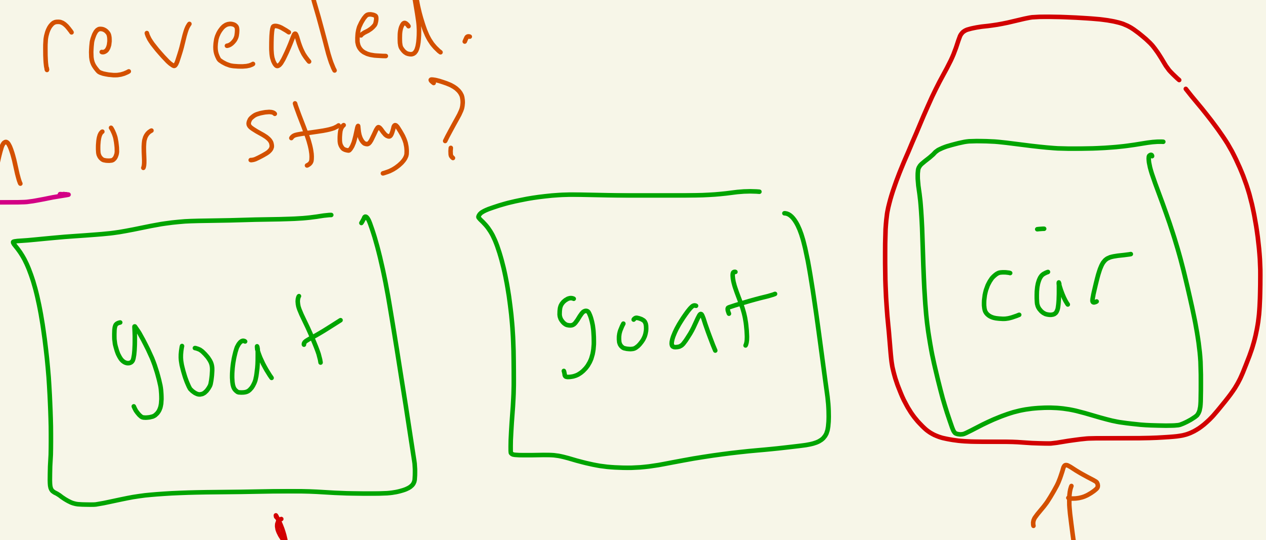
stay with 2 or switch to 3?
MH reveals:

Ex:



door 1 picked.

goat revealed.
switch or stay?



car won!

Analysis

Suppose we always start by picking door 1 and we always switch when given the choice.

Table of possibilities

door 1	door 2	door 3	stay with door 1	switch from door 1
car	goat	goat	WIN	LOSE
goat	car	goat	LOSE	WIN
goat	goat	car	LOSE	WIN

you win 2/3 of the time with the switching strategy

Now on to conditional probability.

Ex: Suppose we roll two 6-sided dice, a green die and a red die. Suppose the green die stops rolling and lands on a 3, but the red die keeps rolling. What's the probability that the sum of the dice is 8?

$\boxed{3}$ $\boxed{?}$

starting sample space

S

(1,1) ⋮ ⋮
(1,2)
(1,3)
⋮

(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)

⋮

(6,5)
(6,6)

36 outcomes

(green, red)

new sample space

S'

(3,1)
(3,2)
(3,3)
(3,4)
(3,5)
(3,6)

6 outcomes

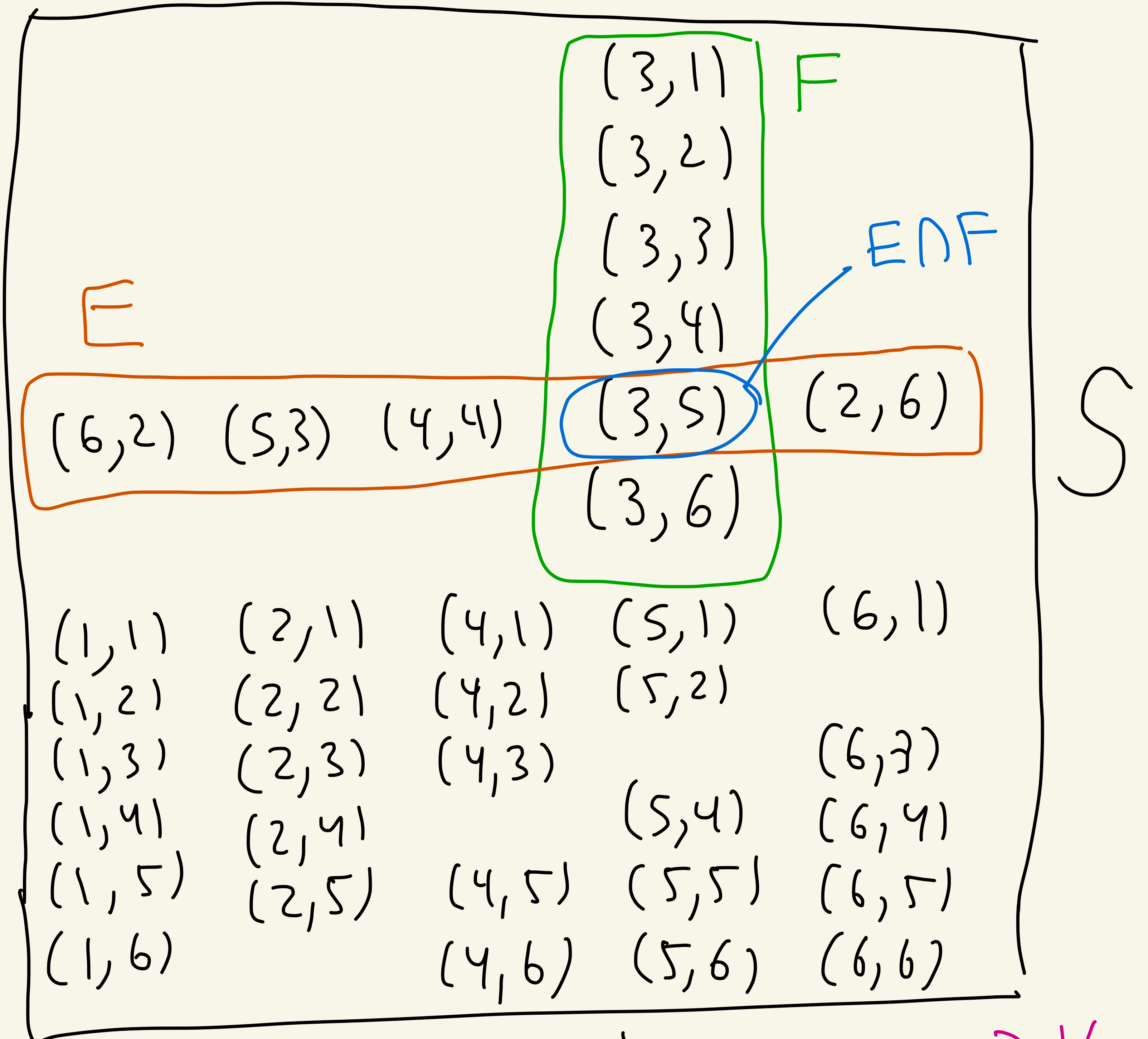
sum is 8

So, the probability is $1/6$

Let's make a formula for this without having to "shrink" the sample space S and also we want it to work for spaces where each outcome is not equally likely.

Let E be the event in S where the sum of the dice is 8.
Let F be the event in S where the green die is 3.
$F = S'$ in above pic.

We want to know the "conditional probability" of the event E occuring given that F has "already occured".



$$\underbrace{\frac{|E\cap F|}{|F|}}_{\text{we calculated this to get } 1/6} = \frac{\left(\frac{|E\cap F|}{|S|}\right)}{\left(\frac{|F|}{|S|}\right)} = \underbrace{\frac{P(E\cap F)}{P(F)}}_{\text{probability in } S} \left\{ \frac{1/36}{6/36} \right.$$

because all outcomes

are equally likely!

Def: Let (S, Ω, P) be a probability space. Let E and F be two events. Suppose $P(F) > 0$. Define the conditional probability that E occurs given that F occured to be

$$\underbrace{P(E \mid F)}_{\text{notation}} = \frac{P(E \cap F)}{P(F)}$$

Ex: (HW 3 #3)

Suppose you roll two 8-sided dice. You can't see the outcome, but your friend can. They tell you that the sum of the dice is divisible by 5. What's the probability that both dice landed on 5?

$$S = \left\{ (a,b) \,\middle|\, \begin{array}{l} a = 1, 2, 3, \ldots, 8 \\ b = 1, 2, 3, \ldots, 8 \end{array} \right\}$$

$$= \{ (1,1), (1,2), \ldots, (8,8) \}$$

$$|S| = 8 \cdot 8 = 64$$

F is event that sum is

divisible by 5.

E is event both dice are 5.

$$F = \{(1,4), (2,3), (2,8), (3,2), (3,7), (4,1), (4,6), (5,5), (6,4), (7,3), (7,8), (8,2), (8,7)\}$$

$$E = \{(5,5)\} \qquad E \cap F = \{(5,5)\}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/64}{13/64}$$

$$= \frac{1}{13}$$

$$\approx \boxed{7.7\%}$$