Math 4740

9/4/24

Ex: Suppose you have a 6-sided die with sides labelled 1,2,3,4,5,6. But through experimentation you realize that each outcome is not equally likely. You estimate the following probabilities:

outcome	probability
1	1/4
2	1/8
3	1/8
4	1/16
5	1/16
6	3/8

Note:

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{3}{8}=1$$

which ought to be.

Let's make a probability space.

$S = \{1, 2, 3, 4, 5, 6\}$

$\Omega = \{\text{all subsets of } S\}$

$= \{\phi, \{1\}, \{2\}, \ldots, \{1,2,4\}, \ldots\}$

We make a probability function

$P : \Omega \to \mathbb{R}$

Define

$P(\{1\}) = 1/4$

$P(\{2\}) = 1/8$

$P(\{3\}) = 1/8$

$P(\{4\}) = 1/16$

$P(\{5\}) = 1/16$

$P(\{6\}) = 3/8$

If E is an event in Ω

define

$$P(E) = \sum_{\omega \in E} P(\{\omega\})$$

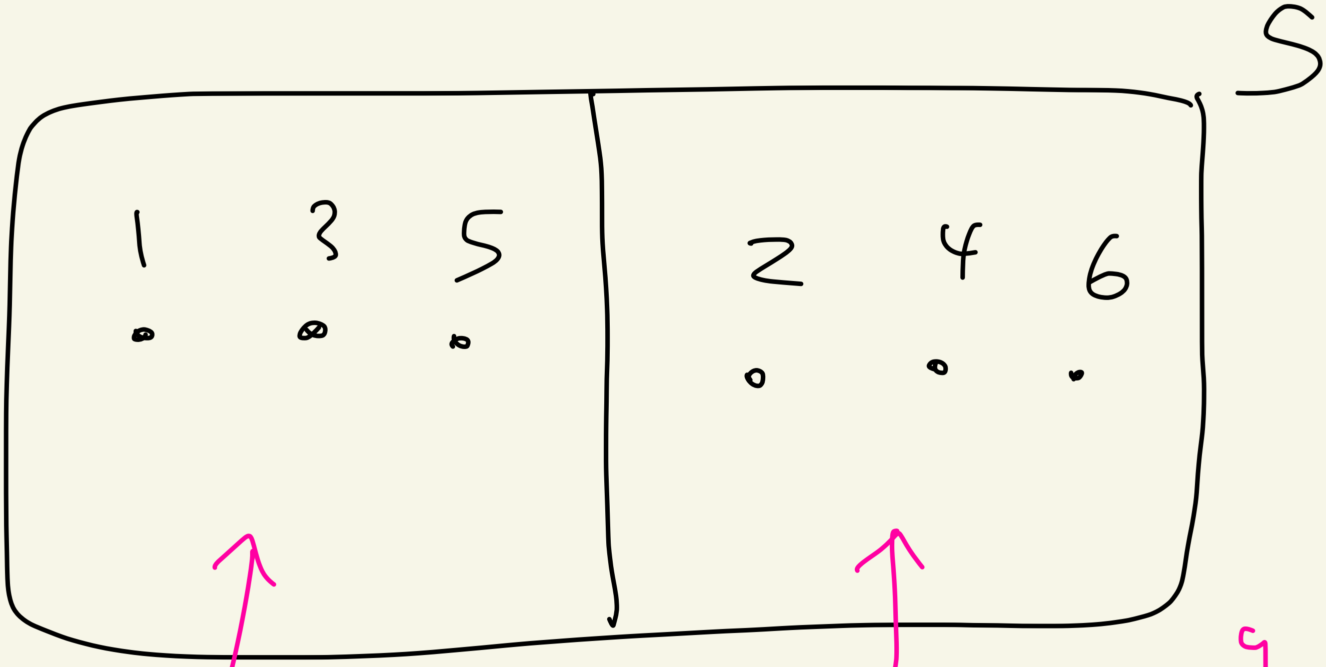
sum over
w in E

For example, what is the probability of rolling an even number?

$$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\})$$
$$= \frac{1}{8} + \frac{1}{16} + \frac{3}{8}$$
$$= \frac{9}{16} = \boxed{0.5625}$$
$$= \boxed{56.25\%}$$

What's the probability of rolling an odd number?

$$P(\{1,3,5\}) = P(\{1\}) + P(\{3\}) + P(\{5\})$$
$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$$
$$= \boxed{0.4375}$$



$P(\{1,3,5\}) = \frac{7}{16}$ $\quad$ $P(\{2,4,6\}) = \frac{9}{16}$

$$\frac{7}{16} + \frac{9}{16} = 1 = P(S)$$

So,

$$P(\{1,3,5\}) = 1 - P(\{2,4,6\})$$
$$= 1 - \frac{9}{16} = \frac{7}{16}$$

<u>Note:</u> Suppose (S, Ω, P) is a probability space and S is finite. Suppose each outcome is equally likely, that is

$$P(\{\omega\}) = \frac{1}{|S|}$$

for all outcomes ω.

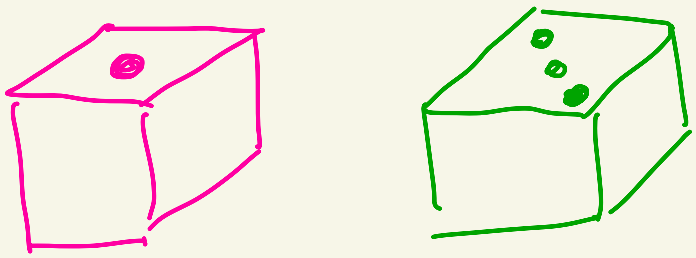
Suppose $E = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is an event.

Then,

$$\begin{aligned} P(E) &= P(\{\omega_1\}) + P(\{\omega_2\}) + \ldots + P(\{\omega_n\}) \\ &= \frac{1}{|S|} + \frac{1}{|S|} + \ldots + \frac{1}{|S|} \\ &= \frac{n}{|S|} = \frac{|E|}{|S|} \end{aligned}$$

So, $P(E) = \dfrac{|E|}{|S|}$ if all outcomes are equally likely.

Ex: Suppose we do the experiment where we roll two 6-sided die. These are normal die, each side is equally likely.



die 1 die 2

encode as $(1,3)$

(a,b) means a on die 1, b on die 2

So,

$$
\begin{aligned}
S = \{ & (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\
& (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\
& (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\
& (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\
& (5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
\end{aligned}
$$

$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$\Omega = \{\text{all subsets of } S\}$

$|S| = 36$

$P(\{(a,b)\}) = \frac{1}{36}$

For ex: $P(\{(2,5)\}) = \frac{1}{36}$

Q: What is the probability that the sum of the dice is 7?

Let E be the event that the sum of the dice is 7.

So,

$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

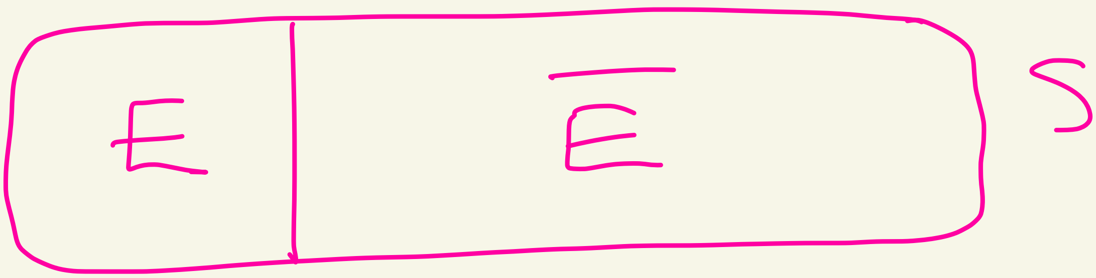
since all outcomes are equally likely

So,

$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$

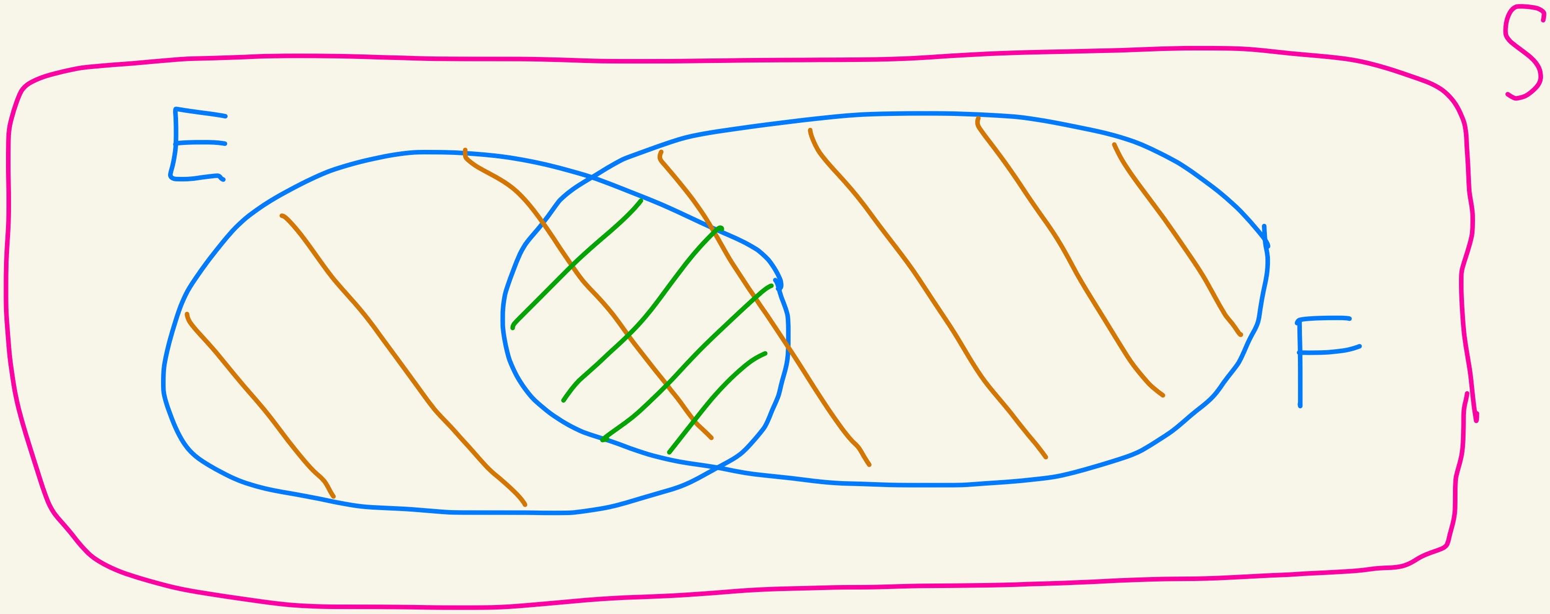
Theorem: Let (S, Ω, P) be a probability space.
Let E and F be events.
Then,

① $P(E) = 1 - P(\overline{E})$



② If $E \subseteq F$, then $P(E) \leq P(F)$

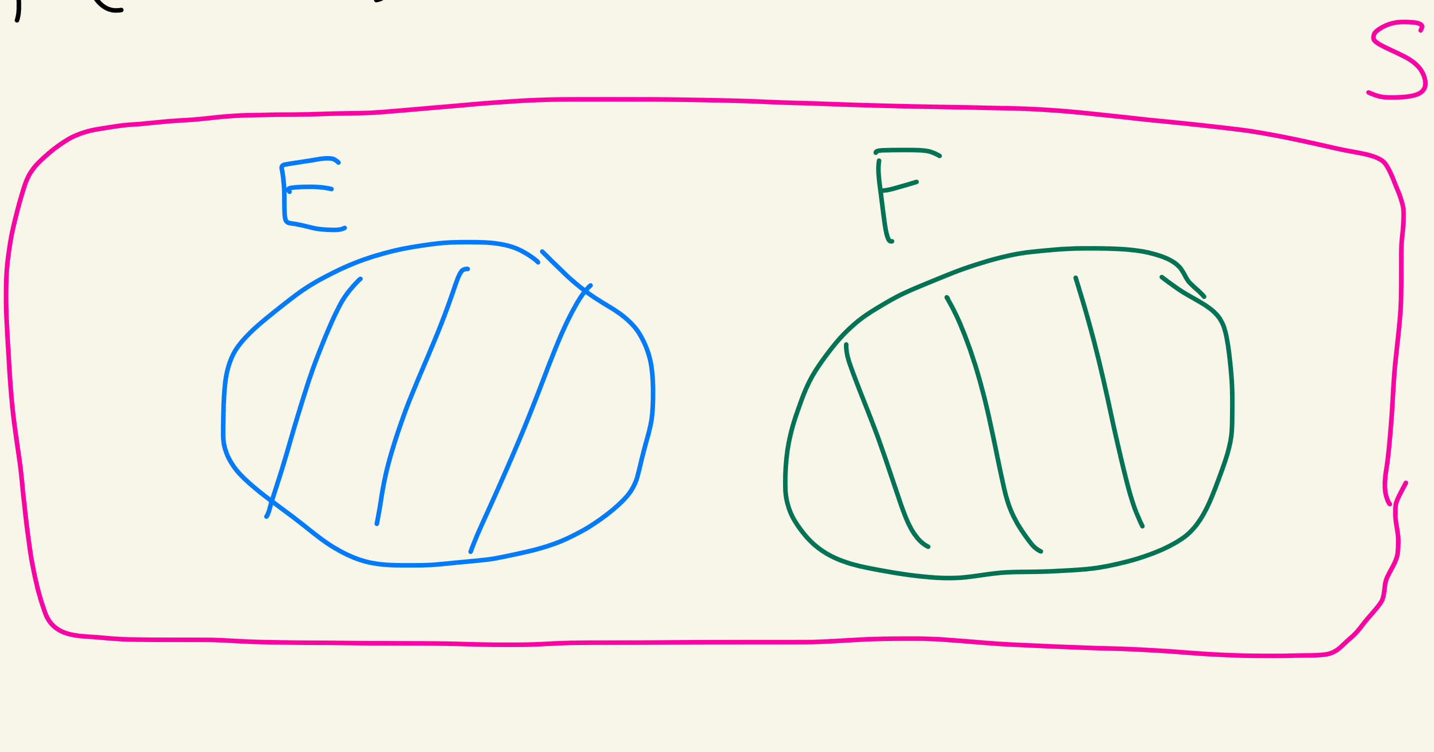
③ $P(E \cup F) = P(E) + P(F) - P(E \cap F)$



$\backslash\backslash\backslash = E \cup F$ $\quad$ $/// = E \cap F$

(4) If E and F are disjoint, that is $E \cap F = \phi$, then

$$P(E \cup F) = P(E) + P(F)$$



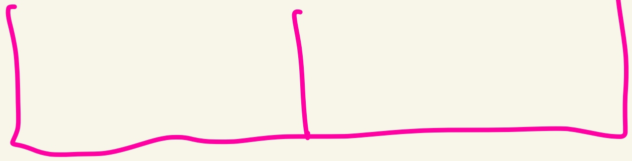
Proof: See online notes. $\blacksquare$

Ex: Suppose we roll two 12-sided dice. The sides are labelled 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Each side is equally likely.

Q: What is the probability that at least one of the dice is greater than or equal to 4?

Ex:

die 1	die 2	
3	2	✗ neither is ≥ 4
2	7	✓ die 2 ≥ 4
11	6	✓ both die



$\geqslant 4$

<u>Sample space:</u>

$$S = \left\{ (a,b) \;\middle|\; \begin{array}{l} a = 1, 2, \dots, 12 \\ b = 1, 2, \dots, 12 \end{array} \right\}$$

$$= \{ (1,1), (1,2), \dots, (12,12) \}$$

Here $|S| = 12 \cdot 12 = 144$

All outcomes are equally likely.

The event E that at least one of the dice is $\geqslant 4$ is

$$E = \{ (a,b) \mid a \geqslant 4 \text{ or } b \geqslant 4 \}$$

$$= \{ (4,1), (4,2), (4,3), \dots, (5,1), \dots \\ (2,10), \dots \}$$

there are too many to count.

Instead let's calculate $\overline{E}$.
Here $\overline{E}$ is both die are < 4.

So,

$$\overline{E} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Then,

$$P(\overline{E}) = \frac{|\overline{E}|}{|S|} = \frac{9}{144} = \frac{1}{16}$$

Thus,

$$P(E) = 1 - P(\overline{E}) = 1 - \frac{1}{16}$$
$$= \frac{15}{16} = \boxed{0.9375}$$
$$= \boxed{93.75\%}$$

Note: You can construct a probability space when S is countably infinite, that is S is infinite and you can list the elements out in an infinite list. That is,

$$S = \{w_1, w_2, w_3, w_4, \ldots\}$$

infinitely many more

Let Ω be the set of all subsets of S.

Define $P(\{w_i\})$ for each i so that $0 \leq P(\{w_i\}) \leq 1$ and

$$\sum_{i=1}^{\infty} P(\{w_i\}) = 1$$

If E is an event define

$$P(E) = \sum_{w \in E} P(\{w\})$$

Theorem: This is a probability space.