Math 4740

Hw 3 Solutions

① $A = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),$
$(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),$
$(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$

$B = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$

$A \cap B = \{(2,1),(2,3),(2,5)\}$

$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$

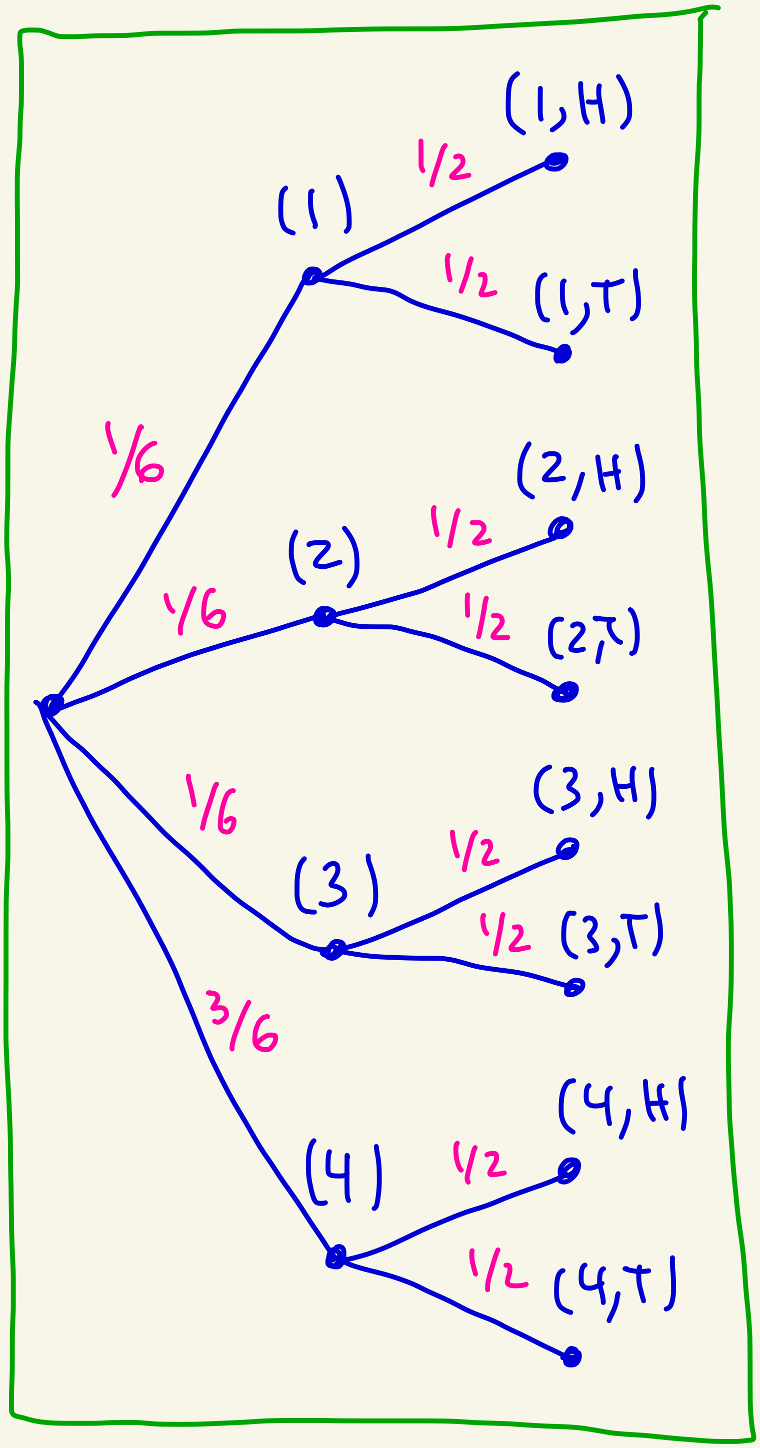
equal

$P(A) \cdot P(B) = \frac{18}{36} \cdot \frac{6}{36} = \frac{1}{12}$

Since $P(A \cap B) = P(A) \cdot P(B)$ the events A and B are independent.

② (a)

$$S = \{(1,H),(2,H),(3,H),(4,H),(1,T),(2,T),(3,T),(4,T)\}$$



multiply branches to get probabilities

$$P(\{(1,H)\}) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$P(\{(1,T)\}) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$P(\{(2,H)\}) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$P(\{(2,T)\}) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$P(\{(3,H)\}) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$P(\{(3,T)\}) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$$

$$P(\{(4,H)\}) = \frac{3}{6}\cdot\frac{1}{2} = \frac{1}{4}$$

$$P(\{(4,T)\}) = \frac{3}{6}\cdot\frac{1}{2} = \frac{1}{4}$$

(b)

$A = \{(1,H),(1,T)\}$

$B = \{(1,H),(2,H),(3,H),(4,H)\}$

$A \cap B = \{(1,H)\}$

$P(A \cap B) = P(\{(1,H)\}) = \frac{1}{12}$

$P(A) = P(\{(1,H)\}) + P(\{(1,T)\}) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

$P(B) = P(\{(1,H)\}) + P(\{(2,H)\})$

$P(\{(3,H)\}) + P(\{(4,H)\})$

$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{4} = \frac{1}{2}$

Thus, $P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$

So, $P(A \cap B) = \frac{1}{12} = P(A) \cdot P(B)$.

Thus, A and B are independent.

3

Let A be the event that the sum of the dice is divisible by 5.

Let B be the event that both dice have landed on 5's.

Then,

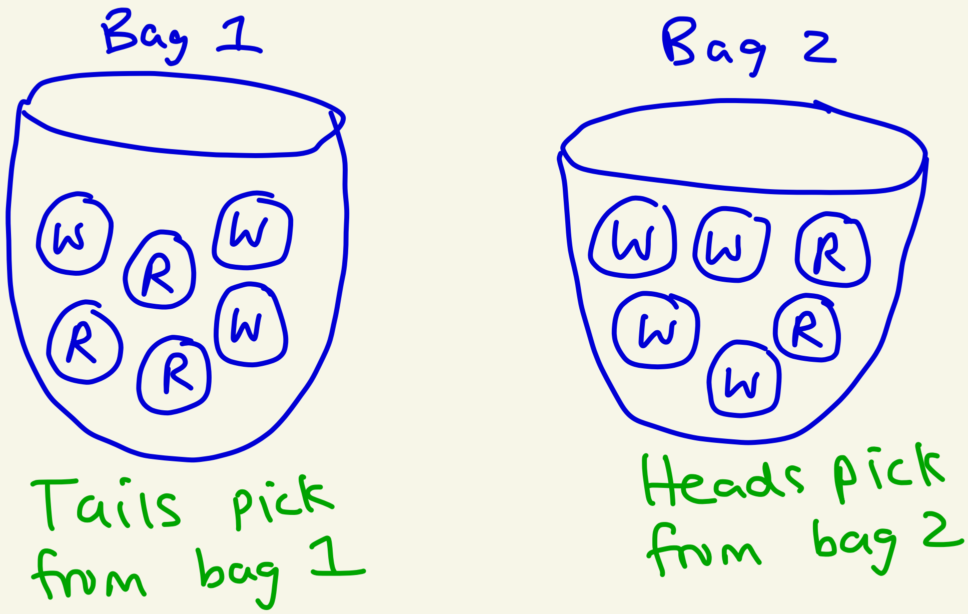
$A = \{(1,4), (2,3), (3,2), (4,1), (4,6), (5,5), (6,4)\}$

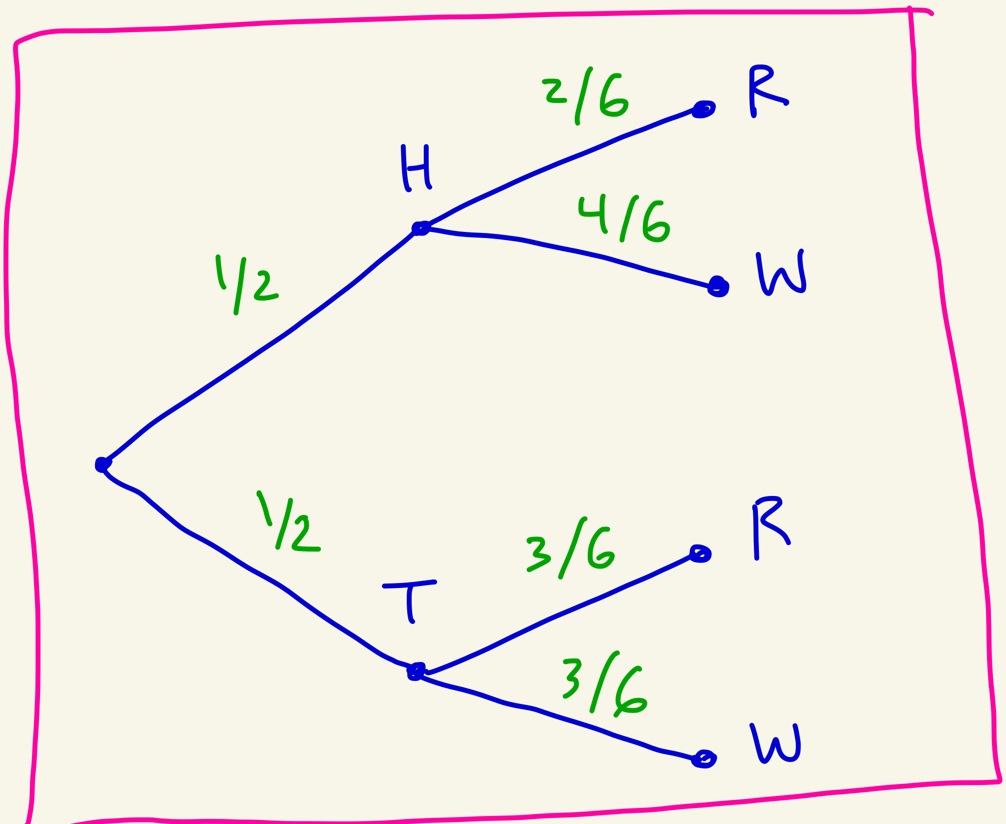
$B = \{(5,5)\}$

$A \cap B = \{(5,5)\}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/36}{7/36} = \frac{1}{7}$$

4

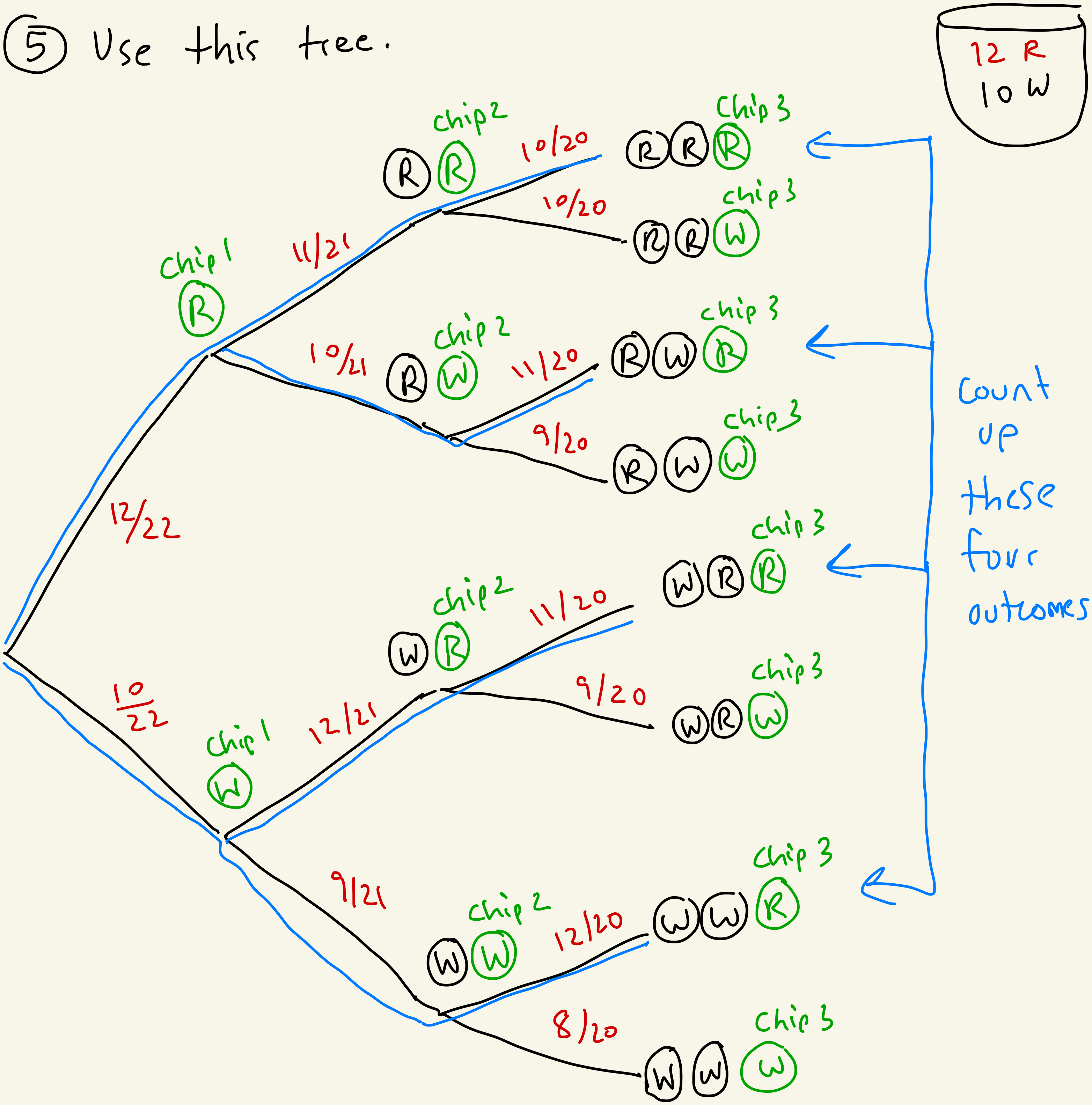




(a) $P(\text{red ball}) = \frac{1}{2}\cdot\frac{2}{6} + \frac{1}{2}\cdot\frac{3}{6} = \frac{2+3}{12} = \frac{5}{12}$

(b) $P(\text{white ball}) = \frac{1}{2}\cdot\frac{4}{6} + \frac{1}{2}\cdot\frac{3}{6} = \frac{4+3}{12} = \frac{7}{12}$

⑤ Use this tree.



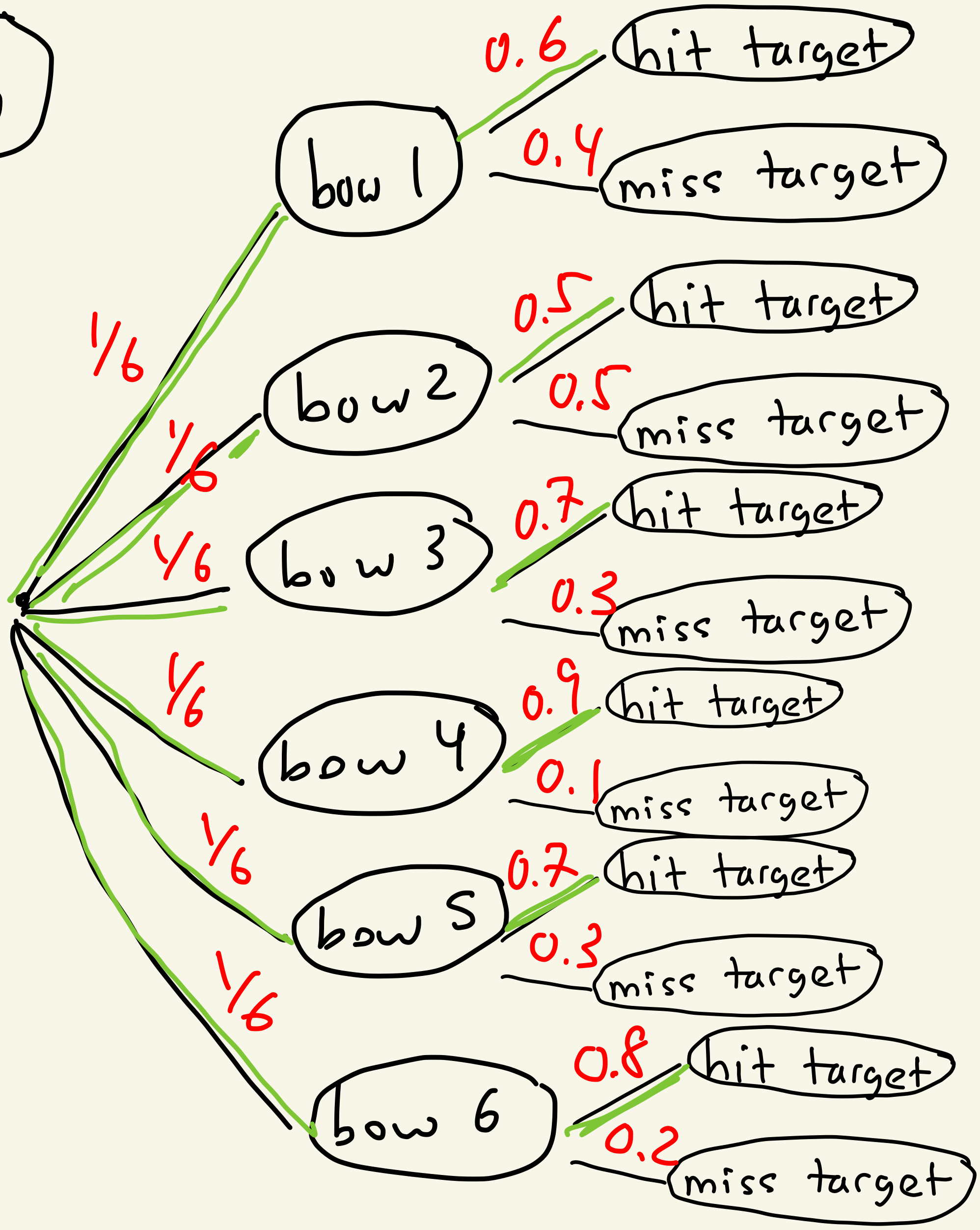
The probability that the 3rd chip is red is

$$\frac{12}{22}\cdot\frac{11}{21}\cdot\frac{10}{20}+\frac{12}{22}\cdot\frac{10}{21}\cdot\frac{11}{20}+\frac{10}{22}\cdot\frac{12}{21}\cdot\frac{11}{20}+\frac{10}{22}\cdot\frac{9}{21}\cdot\frac{12}{20}$$

$$=\boxed{\frac{6}{11}}$$

(add up the blue branches)

6



add up green branches

$$P(\text{hit target}) = \left(\tfrac{1}{6}\right)(0.6) + \left(\tfrac{1}{6}\right)(0.5) + \left(\tfrac{1}{6}\right)(0.7) + \left(\tfrac{1}{6}\right)(0.9) + \left(\tfrac{1}{6}\right)(0.7) + \left(\tfrac{1}{6}\right)(0.8) = \boxed{0.7} = \boxed{70\%}$$

OR

Or you can write it as a formula.

$$
\begin{aligned}
P(\text{hit}) = {} & P(\text{hit} \mid \text{bow 1 picked}) \cdot P(\text{bow 1 picked}) \\
& + P(\text{hit} \mid \text{bow 2 picked}) \cdot P(\text{bow 2 picked}) \\
& + P(\text{hit} \mid \text{bow 3 picked}) \cdot P(\text{bow 3 picked}) \\
& + P(\text{hit} \mid \text{bow 4 picked}) \cdot P(\text{bow 4 picked}) \\
& + P(\text{hit} \mid \text{bow 5 picked}) \cdot P(\text{bow 5 picked}) \\
& + P(\text{hit} \mid \text{bow 6 picked}) \cdot P(\text{bow 6 picked})
\end{aligned}
$$

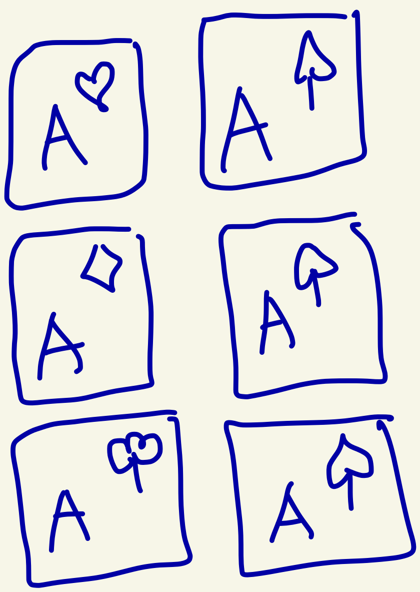
$$
\begin{aligned}
& = (0{,}6)\left(\tfrac{1}{6}\right) + (0{,}5)\left(\tfrac{1}{6}\right) \\
& \quad + (0.7)\left(\tfrac{1}{6}\right) + (0.9)\left(\tfrac{1}{6}\right) \\
& \quad + (0{,}7)\left(\tfrac{1}{6}\right) + (0{,}8)\left(\tfrac{1}{6}\right) \\
& = \boxed{0{,}7} = \boxed{70\%}
\end{aligned}
$$

7 (a)

We will use the formula

$$P(B \mid A_s) = \frac{P(B \cap A_s)}{P(A_s)}$$

$B \cap A_s$ is the event that both of the cards are aces and one of them is the ace of spades. There are 3 ways that this can happen

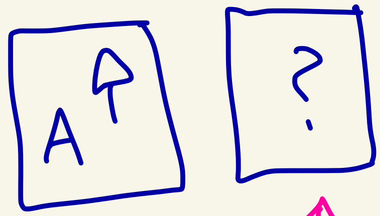


order doesn't matter

There are $\binom{52}{2}$ ways to draw 2 cards from the deck.

Thus, $P(B \cap A_s) = \dfrac{3}{\binom{52}{2}}$

The event A_s is the event that one of the cards is the ace of spades, so your 2 card hand is



some card that isn't A♠
there are 51 choices here

There are 51 of these types of hands.

Thus,

$$P(A_s) = \frac{51}{\binom{52}{2}}$$

So,

$$P(B|A_s) = \frac{P(B \cap A_s)}{P(A_s)} = \frac{3/\binom{52}{2}}{51/\binom{52}{2}} = \frac{3}{51} = \frac{1}{17}$$

$$\boxed{\begin{aligned} &\approx 0.0588 \\ &\approx 5.88\% \end{aligned}}$$

7 (b)

We use $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$

Note that $B \cap A = B$

(both cards are aces) and (at least one of the cards is an ace)

(both cards are aces)

There are $\binom{4}{2} = 6$ ways to get 2 aces

[Since there are 4 total aces]

[Note you can list them actually

A♥ A♣, A♥ A♦, A♥ A♠, A♣ A♦

A♣ A♠, A♦ A♠]

Thus, $P(B \cap A) = P(B) = \dfrac{\binom{4}{2}}{\binom{52}{2}} = \dfrac{6}{\binom{52}{2}}$

$P(A)$ is the probability that at least one of the cards is an ace.

Let's instead do $P(\overline{A})$ which is the probability that neither card is an ace.

There are $52-4=48$ cards that aren't aces.

Thus, $P(\overline{A}) = \dfrac{\binom{48}{2}}{\binom{52}{2}}$

So, $P(A) = 1 - P(\overline{A}) = 1 - \dfrac{\binom{48}{2}}{\binom{52}{2}}$

Note $\binom{48}{2} = \dfrac{48!}{2!\,46!} = \dfrac{48 \cdot 47}{2} = 1128$

and $\binom{52}{2} = \dfrac{52!}{2!\,50!} = \dfrac{52 \cdot 51}{2} = 1326$

Thus,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)}$$

$$= \frac{6/\binom{52}{2}}{1 - \frac{\binom{48}{2}}{\binom{52}{2}}} = \frac{\left(6/1326\right)}{\left(1 - \frac{1128}{1326}\right)}$$

$$= \frac{1326}{1326} \cdot \frac{\left(6/1326\right)}{\left(1 - \frac{1128}{1326}\right)} = \frac{6}{1326 - 1128}$$

$$= \frac{6}{198} = \frac{1}{33} \boxed{\approx 0.03\overline{03} \approx 3.03\%}$$

(8) Let S be the sample space of drawing 10 cards one by one from a 52 card deck. Here order matters

___	___	___	___	___	___	___	___	___	___
↑ 52 possibilities	↑ 51	↑ 50	↑ 49	↑ 48	↑ 47	↑ 46	↑ 45	↑ 44	↑ 43

$$|S| = 52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43$$

Let F be the event that there are exactly 3 hearts in the first nine spots.

Step 1: Choose 3 out 9 spots for the hearts.

$\boxed{\underset{1}{\_}\ \ \overset{\heartsuit}{\underset{2}{\_}}\ \ \overset{\heartsuit}{\underset{3}{\_}}\ \ \underset{4}{\_}\ \ \underset{5}{\_}\ \ \overset{\heartsuit}{\underset{6}{\_}}\ \ \underset{7}{\_}\ \ \underset{8}{\_}\ \ \underset{9}{\_}}\ \ \underset{10}{\_}$

$$\binom{9}{3} = \frac{9!}{6!\,3!} = \frac{9\cdot 8\cdot 7}{6} = 3\cdot 4\cdot 7 = 84$$

Step 2: put the face values into the heart spots

	K♥	5♥			2♥				
1	2	3	4	5	6	7	8	9	10
	13 possibilities	12 possibilities			11 possibilities				

Step 3: Fill spots 1 – 9 with non-hearts

There are $13 \cdot 3 = 39$ non-hearts

J♦	K♥	5♥	Q♠	Q♣	2♥	3♦	2♦	3♠	
1	2	3	4	5	6	7	8	9	10
39 possibilities			38	37		36	35	34	

Step 4: Fill in the 10^{th} card that can be anything. There are $52 - 9 = 43$ cards left.

J♦	K♥	5♥	Q♠	Q♣	2♥	3♦	2♦	3♠	10♣
1	2	3	4	5	6	7	8	9	10
									43 possibilities

So,

$$P(F) = \frac{\binom{9}{3} \cdot 13 \cdot 12 \cdot 11 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 43}{|S|}$$

$$= \frac{84 \cdot 13 \cdot 12 \cdot 11 \cdot 39 \cdot 38 \cdot 37 \cdot 36 \cdot 35 \cdot 34 \cdot 43}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}$$

$$= \frac{27{,}417}{108{,}100}$$

Now let's calculate $P(E|F)$. We are given that there are exactly 3 hearts in the first 9 cards (so 6 non-hearts in the first 9 cards). We want to know the chances of getting a heart on the 10^{th} card given this

$\underline{3^\diamondsuit}$ $\underline{K^\heartsuit}$ $\underline{10^\spadesuit}$ $\underline{K^\diamondsuit}$ $\underline{Q^\heartsuit}$ $\underline{3^\heartsuit}$ $\underline{K^\clubsuit}$ $\underline{7^\diamondsuit}$ $\underline{3^\spadesuit}$ $\underline{\quad}$

10 possibilities

There are 10 possible hearts and $52-9=43$ cards to choose from. Thus,

$$P(E|F) = \frac{10}{43}$$

Thus,

$$P(E \cap F) = P(F) \cdot P(E|F)$$

$$= \left(\frac{27,417}{108,100}\right)\left(\frac{10}{43}\right)$$

$$\approx 0.05898\ldots$$

$$\approx 5.9\,\%$$

9) Let S be the sample space for rolling two 6-sided die. Let A be the event that the sum of the dice is 6. Let B be the event that the sum of the dice is 7.

Then A and B are disjoint.

Suppose we repeat S over and over until either A or B occurs.

(a) The probability that you roll a sum of 6 before rolling a sum of 7 is

$$\frac{P(A)}{P(A)+P(B)} = \frac{5/36}{5/36+6/36} = \frac{5}{11}$$

(b) The probability that you roll a sum of 7 before rolling a sum of 6 is

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36+5/36} = \frac{6}{11}$$

(10) Let S be the sample space of selecting one card from a standard 52-card deck. Let A be the event that the card selected is an ace. Let B be the event that the card selected is a face card.

Then A and B are disjoint.

We repeat S over and over until either A or B occurs.

Recall: There are 4 aces and 12 face cards (4 jacks + 4 queens + 4 kings)

(a) The probability that an ace comes up before a face card is

$$\frac{P(A)}{P(A)+P(B)} = \frac{4/52}{4/52 + 12/52} = \frac{4}{16} = \frac{1}{4}$$

(b) The probability that a face card comes up before an ace is

$$\frac{P(B)}{P(B)+P(A)} = \frac{12/52}{12/52 + 4/52} = \frac{12}{16} = \frac{3}{4}$$

(11) Let RR, BB, and RB denote, respectively, the events that the chosen card is all red, all black, or the red-black card. Let R be the event that after we randomly choose a card and put it down on the ground the up-side is red. We want $P(RB \mid R)$. We have:

$$\boxed{P(RB \mid R) = \frac{P(RB \cap R)}{P(R)}} \quad (*)$$

We can write the numerator of (*) as

$$P(RB \cap R) = P(R \mid RB) \cdot P(RB)$$

Since

$$P(R \mid RB) = \frac{P(R \cap RB)}{P(RB)}$$

This becomes

$$P(RB \cap R) = P(R \mid RB) \cdot P(RB)$$
$$= \left(\tfrac{1}{2}\right) \cdot \left(\tfrac{1}{3}\right) = \frac{1}{6}$$

For the denominator of $(*)$ we use the law of total probability to get

$$\begin{aligned} P(R) = {} & P(R/RR) \cdot P(RR) \\ & + P(R/RB) \cdot P(RB) \\ & + P(R \mid BB) \cdot P(BB) \end{aligned}$$

This can also be thought of as a tree see next page

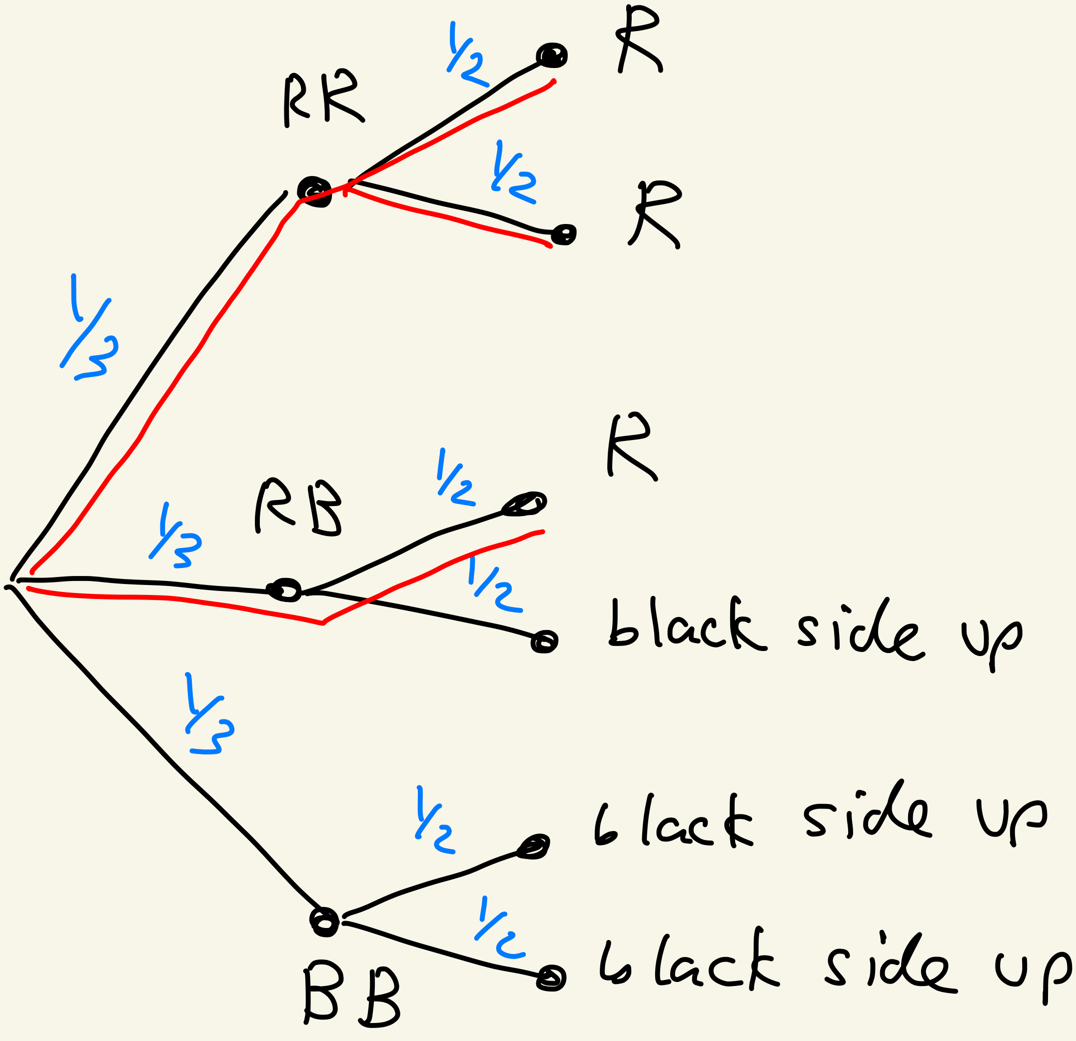
$$= (1)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) + (0)\left(\tfrac{1}{3}\right)$$
$$= \frac{1}{2}$$

Therefore, $(*)$ becomes

$$P(RB \mid R) = \frac{P(RB \cap R)}{P(R)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Tree for:

$$P(R) = P(R/RR) \cdot P(RR) + P(R/RB) \cdot P(RB) + P(R|BB) \cdot P(BB)$$



Adding up endpoints that are red:

$$\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}$$

(12) Let BB, BR, and RR be the events that the discarded balls are blue and blue, blue and red, or red and red, respectively.

Let R be the event that the third ball is red.

Then,

$$\boxed{P(BB \mid R) = \frac{P(BB \cap R)}{P(R)}} \quad (*)$$

The numerator of $(*)$ becomes

$$P(BB \cap R) = P(R \mid BB) \cdot P(BB)$$

Since $P(R \mid BB) = \dfrac{P(BB \cap R)}{P(BB)}$

This gives

$$P(BB \cap R) = P(R \mid BB) \cdot P(BB)$$
$$= \left(\frac{7}{18}\right)\left(\frac{13 \cdot 12}{20 \cdot 19}\right) = \frac{91}{570}$$

We can use the law of total probability of (*) to to deal with the denominator to get that

$$\boxed{P(R) = P(R \mid BB) \cdot P(BB) + P(R \mid BR) \cdot P(BR) + P(R \mid RR) \cdot P(RR)} \quad (**)$$

Note $P(BB) = \frac{13 \cdot 12}{20 \cdot 19} = \frac{39}{95}$, $P(RR) = \frac{7 \cdot 6}{20 \cdot 19} = \frac{21}{190}$,

$$P(BR) = \frac{13 \cdot 7}{20 \cdot 19} + \frac{7 \cdot 13}{20 \cdot 19} \leftarrow \boxed{\text{First blue then red or first red then blue}}$$
$$= \frac{91}{190}$$

Thus, (**) gives

$$P(R) = \left(\frac{7}{18}\right)\left(\frac{39}{95}\right) + \left(\frac{6}{18}\right)\left(\frac{91}{190}\right) + \left(\frac{5}{18}\right)\left(\frac{21}{190}\right)$$
$$= 7/20$$

Putting this all together (*) becomes

$$P(BB \mid R) = \frac{P(BB \cap R)}{P(R)}$$

$$= \frac{91/570}{7/20}.$$

$$= \frac{26}{57}$$

$$\approx \boxed{0.45614}$$

$$\approx \boxed{45.6\%}$$