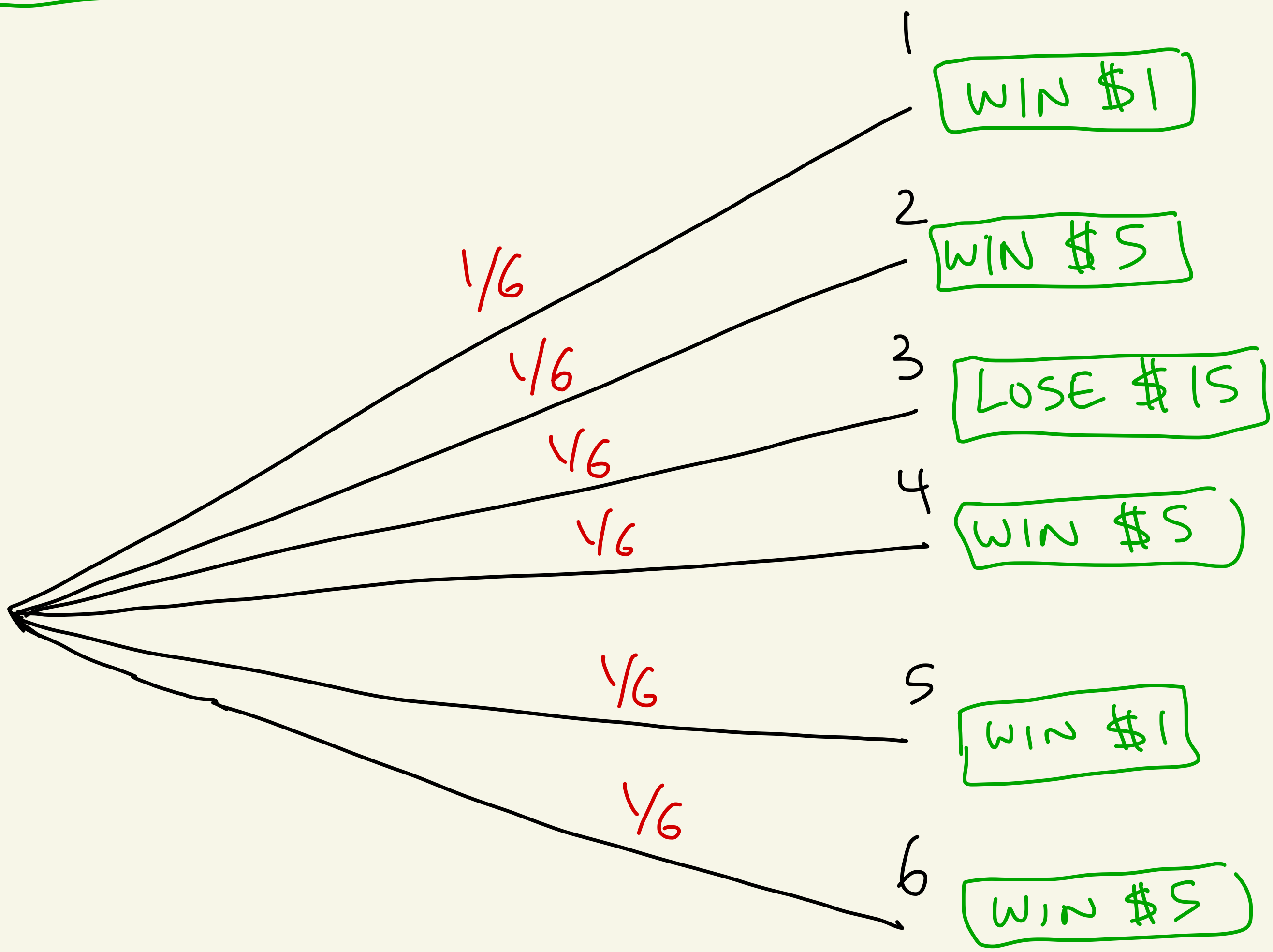
Math 4740

HW 4 Solutions

① (a) Let's make the tree of all outcomes.



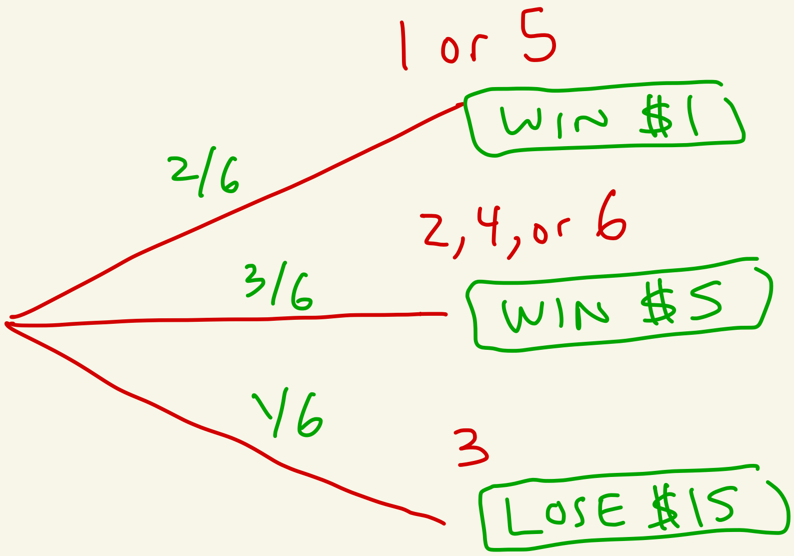
Let X be the amount won or lost.

$$E[X] = (\$1)\underbrace{\left(\frac{1}{6}+\frac{1}{6}\right)}_{\text{probability you win \$1}} + (\$5)\underbrace{\left(\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\right)}_{\text{probability you win \$5}}$$

$$+(-\$15)\underbrace{\left(\frac{1}{6}\right)}_{\text{probability you lose \$15}} = \$\,\frac{2+15-15}{6} = \$\,\frac{1}{3}$$

$$\approx \boxed{\$0.33\ldots}$$

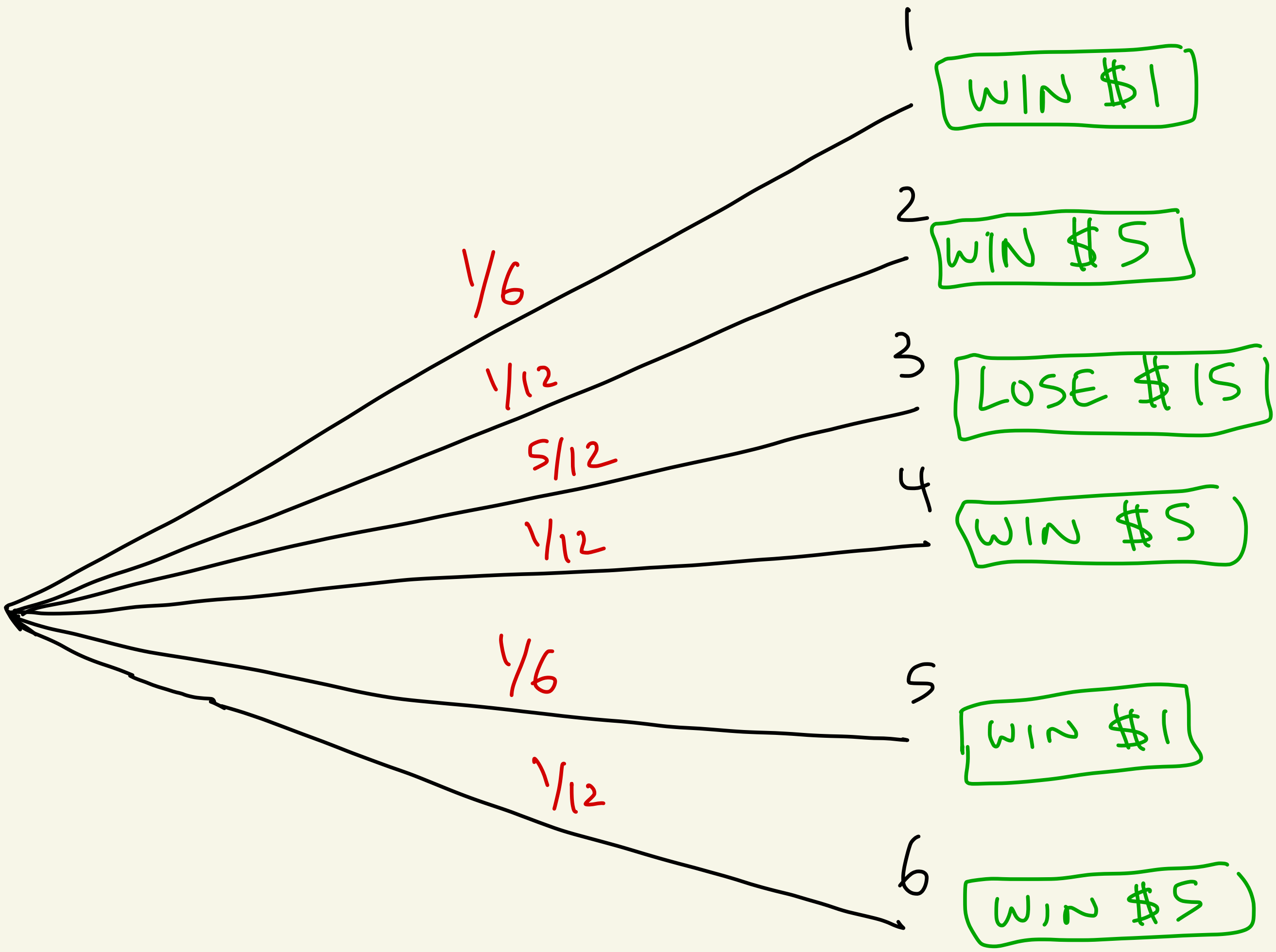
You could have alternatively merged some of the branches like this:



Then: $E[X] = (\$1)\left(\frac{2}{6}\right) + (\$5)\left(\frac{3}{6}\right) + (-\$15)\left(\frac{1}{6}\right)$

$$= \$\,\frac{2+15-15}{6} \approx \$0.33$$

(1)(b) Let's make the tree of all outcomes.



Let X be the amount won or lost.

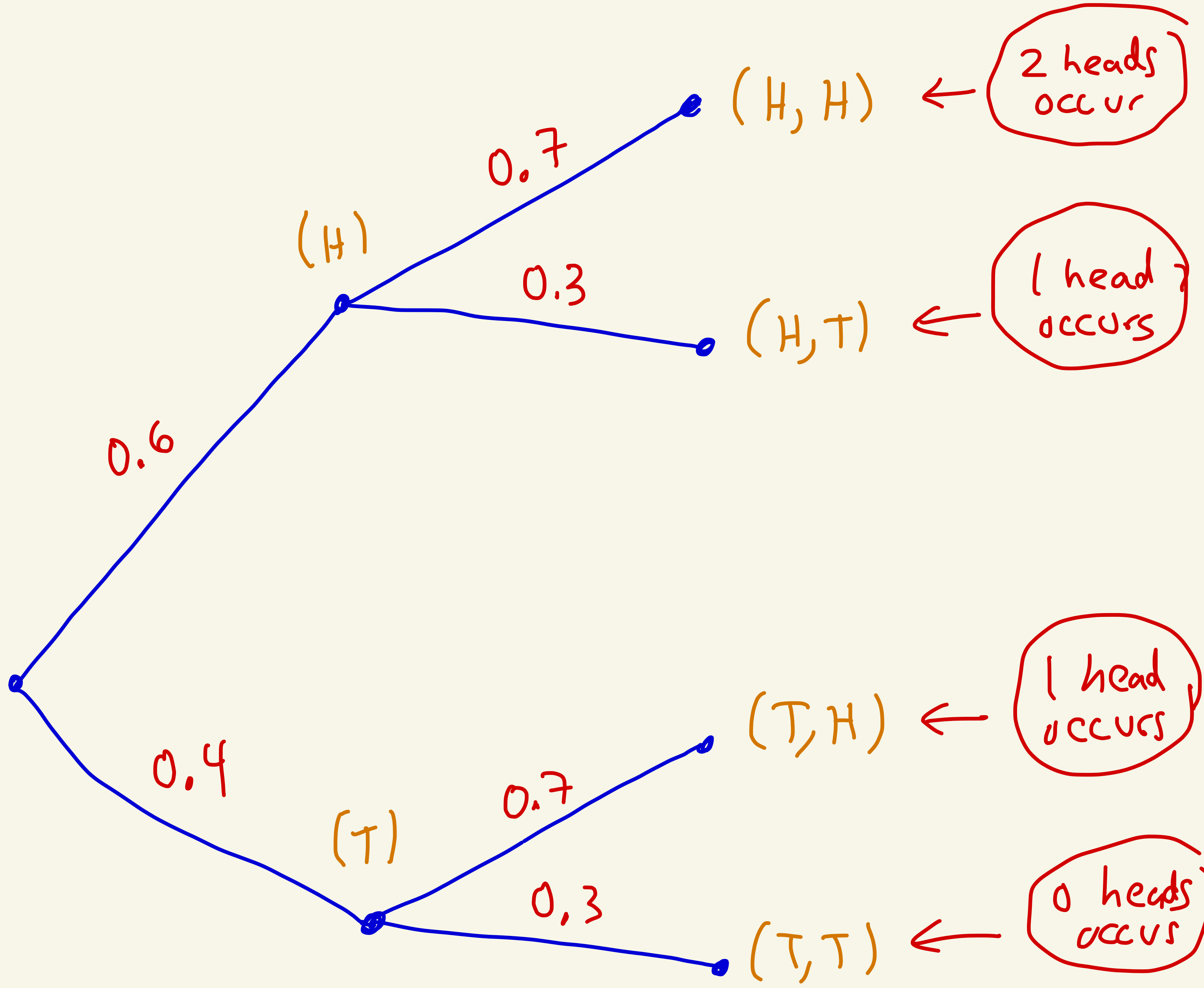
$$E[X] = (\$1)\underbrace{\left(\frac{1}{6}+\frac{1}{6}\right)}_{\text{probability you win \$1}} + (\$5)\underbrace{\left(\frac{1}{12}+\frac{1}{12}+\frac{1}{12}\right)}_{\text{Probability you win \$5}}$$

$$+ (-\$15)\underbrace{\left(\frac{5}{12}\right)}_{\text{probability you lose \$15}} = \$\,\frac{2+15-75}{12} = \boxed{-\$\frac{58}{12}}$$

$$\approx \boxed{-\$4.83\ldots}$$

2 Let $S = \{(H,H), (H,T), (T,H), (T,T)\}$

where (a,b) means a is the result of coin A and b is the result of coin B.



$\boxed{(a)}$ X = # heads that occur

$$P(X=0) = P(\{(T,T)\}) = (0.4)(0.3) = 0.12$$

$$P(X=1) = P(\{(H,T),(T,H)\}) = (0.6)(0.3) + (0.4)(0.7) = 0.18 + 0.28 = 0.46$$

$$P(X=2) = P(\{(H,H)\}) = (0.6)(0.7) = 0.42$$

$\boxed{(b)}$
$$E[X] = (0)(0.12) + (1)(0.46) + (2)(0.42) = 0 + 0.46 + 0.84 = 1.3$$

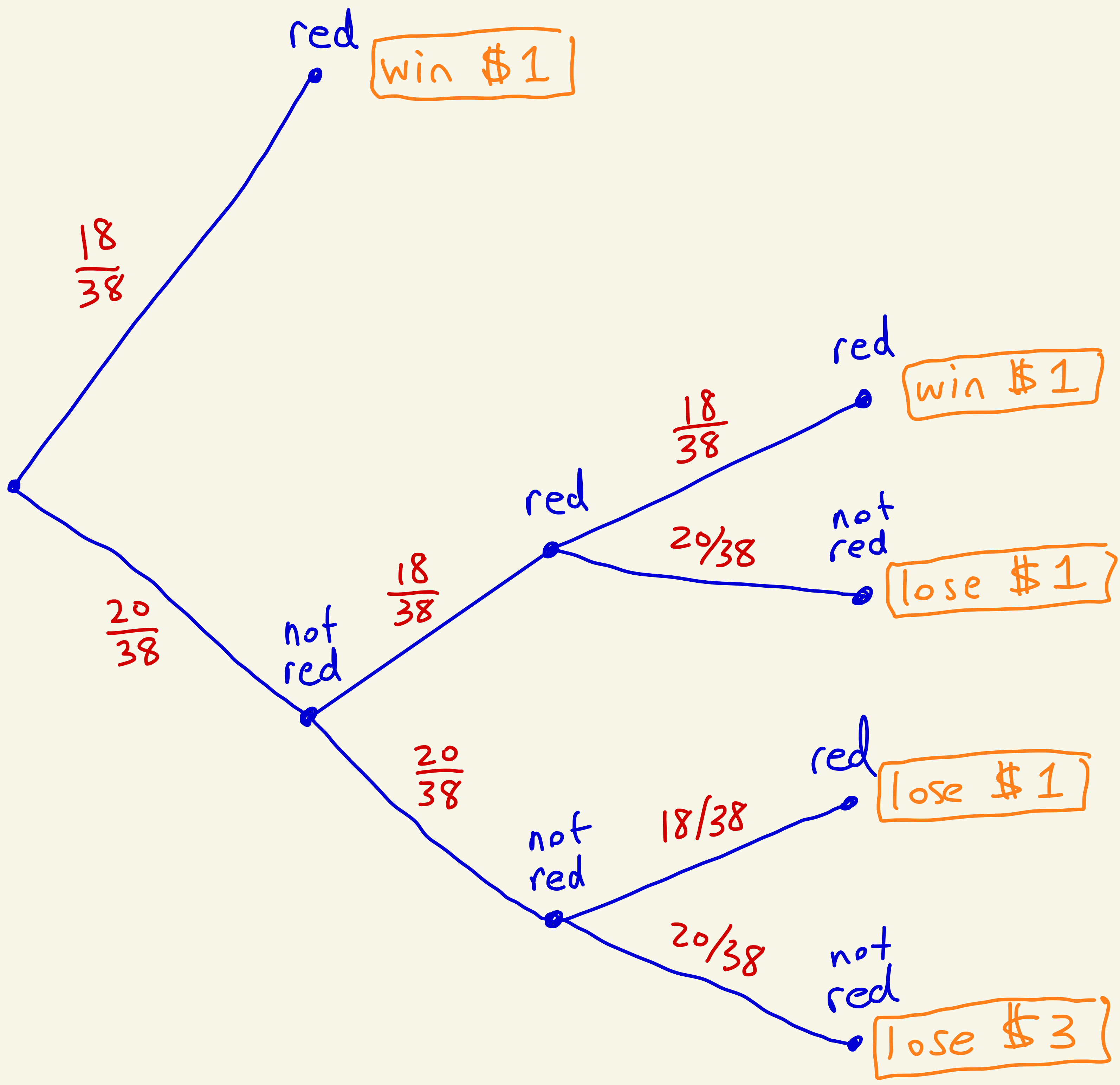
Thus on average 1.3 heads occur on each experiment over the long term.

3 Let's draw the possibilities using a tree.

Probability of getting red: $\frac{18}{38}$

probability of getting non-red: $\frac{20}{38}$

Roulette:
18 reds
18 blacks
+ 2 greens
38 total



(a) $P(X>0)$ means the probability that you win something since X is the amount won or lost.

Thus,

$$P(X>0) = \underbrace{\frac{18}{38}}_{\substack{\text{top branch}\\ \text{of tree}\\ \text{leading to}\\ \text{win \$1}}} + \underbrace{\frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38}}_{\substack{\text{second branch of}\\ \text{tree leading to}\\ \text{win \$1}}} = \boxed{\frac{4059}{6859}} \approx \boxed{0.5917\ldots}$$

(b) $$E[X] = (\$1)\cdot\underbrace{\left[\frac{18}{38} + \frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38}\right]}_{\text{probability you win \$1}}$$

$$+ (-\$1)\cdot\underbrace{\left[\frac{20}{38}\cdot\frac{18}{38}\cdot\frac{20}{38} + \frac{20}{38}\cdot\frac{20}{38}\cdot\frac{18}{38}\right]}_{\text{probability you lose \$1}}$$

$$+ (-\$3)\cdot\underbrace{\left[\frac{20}{38}\cdot\frac{20}{38}\cdot\frac{20}{38}\right]}_{\text{probability you lose \$3}}$$

$$= \boxed{-\$\frac{39}{361}} \approx \boxed{-\$0.11}$$

So even though the probability that we win is about 59%, we lose on average about \$0.11 per game.

4 (a)

Initially you are dealt:

5 cards given

deck

4♥, 10♥, Q♥, 3♠, 2♣

starts with 52 cards

Note you got 3 hearts.

Now you get rid of 3♠, 2♣ and get two new cards from the deck

2 more cards given

deck

4♥, 10♥, Q♥, ?, ?

has:

$47 = 52 - 5$ cards

$10 = 13 - 3$ hearts

Thus, the probability you got 2 more hearts is:

choose 2 hearts from 10

choose 2 cards from 47

$$\frac{\binom{10}{2}}{\binom{47}{2}} = \frac{\left(\frac{10!}{2!\,8!}\right)}{\left(\frac{47!}{2!\,45!}\right)} = \frac{\left(\frac{10\cdot 9\cdot \cancel{8!}}{\cancel{2}\cdot \cancel{8!}}\right)}{\left(\frac{47\cdot 46\cdot \cancel{45!}}{\cancel{2}\cdot \cancel{45!}}\right)} = \frac{10\cdot 9}{47\cdot 46}$$

$$= \boxed{\frac{45}{1081}} \approx \boxed{0.0416} \approx \boxed{4.16\%}$$

4 (b) Let X = amount won or lost

$E[X] =$

$$= (\$500)\left[\begin{array}{c}\text{probability}\\ \text{you get}\\ \text{a flush}\end{array}\right] + (-\$20)\left[\begin{array}{c}\text{probability you}\\ \text{don't get a}\\ \text{flush}\end{array}\right]$$

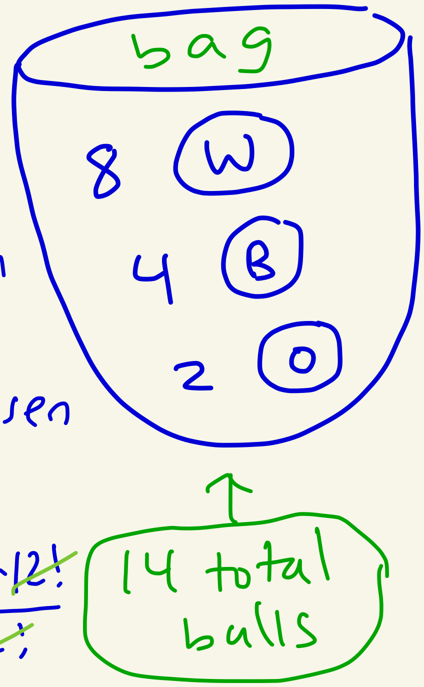
$$= (\$500)\underbrace{\left(\frac{45}{1081}\right)}_{\text{from part (a)}} + (-\$20)\left[1 - \frac{45}{1081}\right]$$

$$= \boxed{\$\frac{1780}{1081}} \approx \boxed{\$1.6466\ldots} \approx \boxed{\$1.65}$$

This is a good bet if you can play the game many times since on average over many plays you'd win \$1.65 per game.

5 Define:

WW – event 2 white balls chosen
WB – event 1 white / 1 black balls chosen
WO – event 1 white / 1 orange balls chosen
BB – event 2 black balls chosen
BO – event 1 black / 1 orange balls chosen
OO – event 2 orange balls chosen



Sample space size is $\binom{14}{2} = \frac{14!}{2!\,12!} = \frac{14\cdot 13\cdot \cancel{12!}}{2\cdot \cancel{12!}}$

$= 91$

Also, $P(WW) = \dfrac{\binom{8}{2}}{\binom{14}{2}} = \dfrac{28}{91}$

$P(WO) = \binom{8}{1}\binom{2}{1} \Big/ \binom{14}{2} = \dfrac{16}{91}$

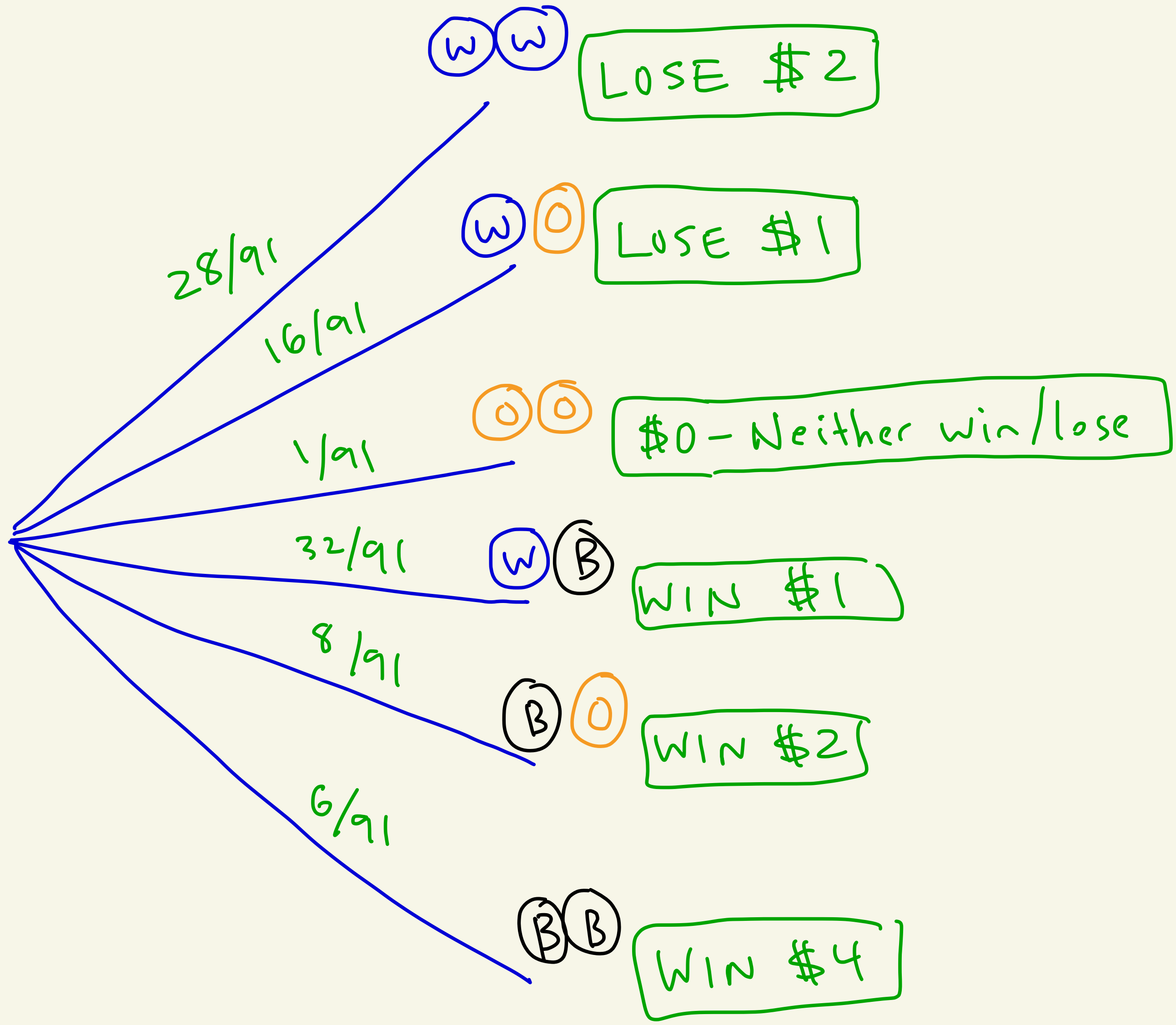
$P(OO) = \binom{2}{2} \Big/ \binom{14}{2} = \dfrac{1}{91}$

$P(WB) = \binom{8}{1}\binom{4}{1} \Big/ \binom{14}{2} = \dfrac{32}{91}$

$P(BO) = \binom{4}{1}\binom{2}{1} \Big/ \binom{14}{2} = \dfrac{8}{91}$

$P(BB) = \binom{4}{2} \Big/ \binom{14}{2} = \dfrac{6}{91}$

The tree is:



X = amount won or lost

$$E[X] = (-\$2)\left(\frac{28}{91}\right) + (-\$1)\left(\frac{16}{91}\right) + (\$0)\left(\frac{1}{91}\right)$$
$$+ (\$1)\left(\frac{32}{91}\right) + (\$2)\left(\frac{8}{91}\right) + (\$4)\left(\frac{6}{91}\right)$$

$$= \boxed{\$0}$$

6

(a) Size of sample space $|S| = 6^3$

You lose $-\$1$ if none of the dice match your number. Thus,

$$\boxed{p(-1) = P(X=-1) = \frac{5 \cdot 5 \cdot 5}{6^3} = \frac{125}{216}}$$

You win $\$1$ if exactly one die matches your number. Thus,

$$\boxed{p(1) = P(X=1) = \frac{1 \cdot 5 \cdot 5 + 5 \cdot 1 \cdot 5 + 5 \cdot 5 \cdot 1}{6^3} = \frac{75}{216}}$$

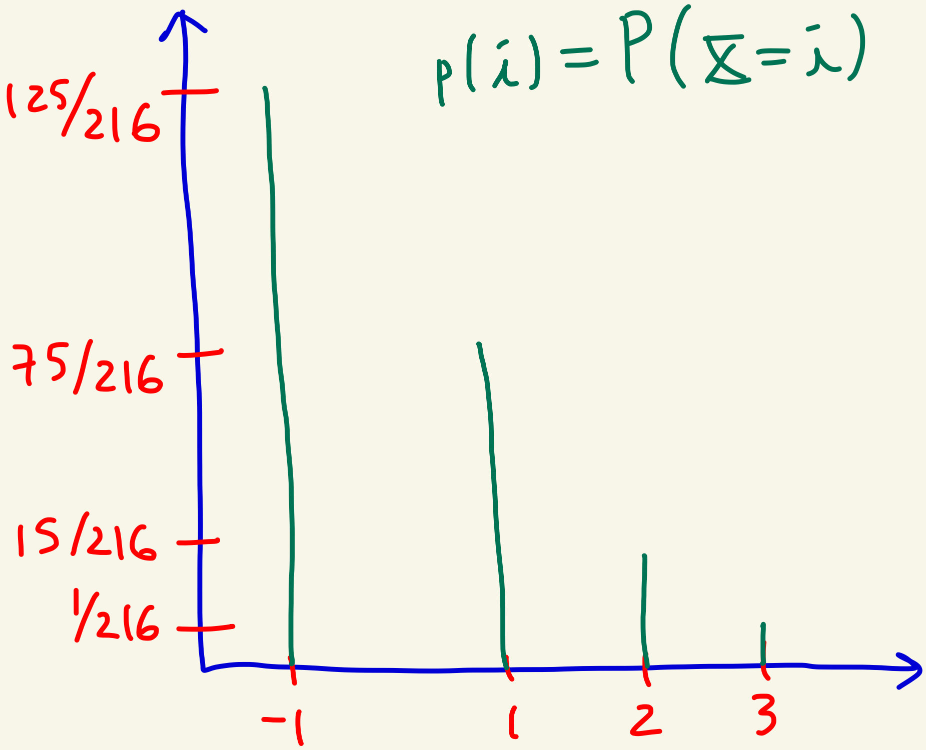
You win $\$2$ if exactly two dice match your number. Thus,

$$\boxed{p(2) = P(X=2) = \frac{1 \cdot 1 \cdot 5 + 1 \cdot 5 \cdot 1 + 5 \cdot 1 \cdot 1}{6^3} = \frac{15}{216}}$$

You win $\$3$ if all the dice match your number. Thus,

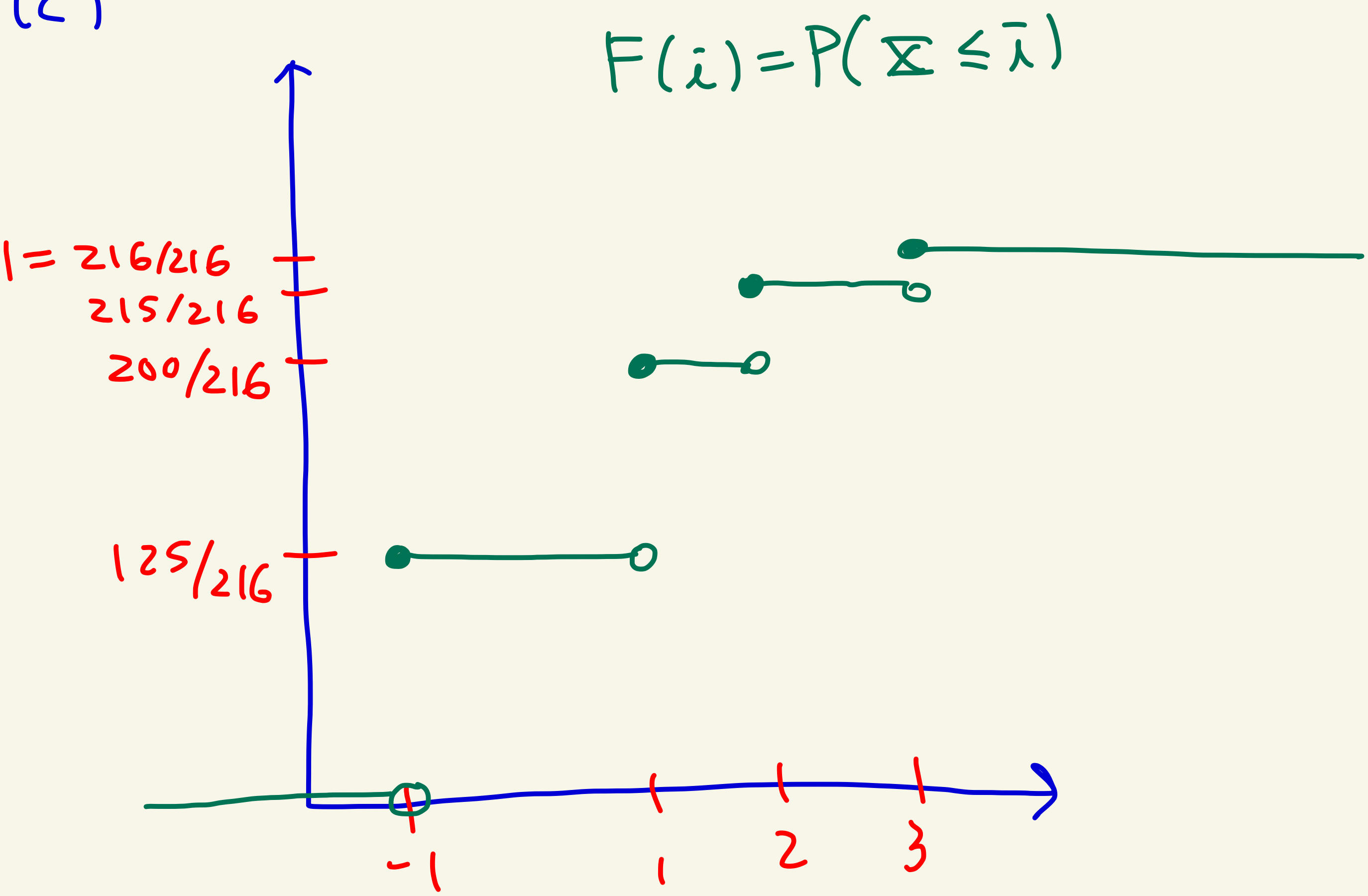
$$\boxed{p(3) = P(X=3) = \frac{1 \cdot 1 \cdot 1}{6^3} = \frac{1}{6^3} = \frac{1}{216}}$$

(b)



$p(i) = P(X = i)$
125/216
75/216
15/216
1/216
-1
1
2
3

(c)



$F(i) = P(X \leq i)$
1 = 216/216
215/216
200/216
125/216
-1
1
2
3

(d)

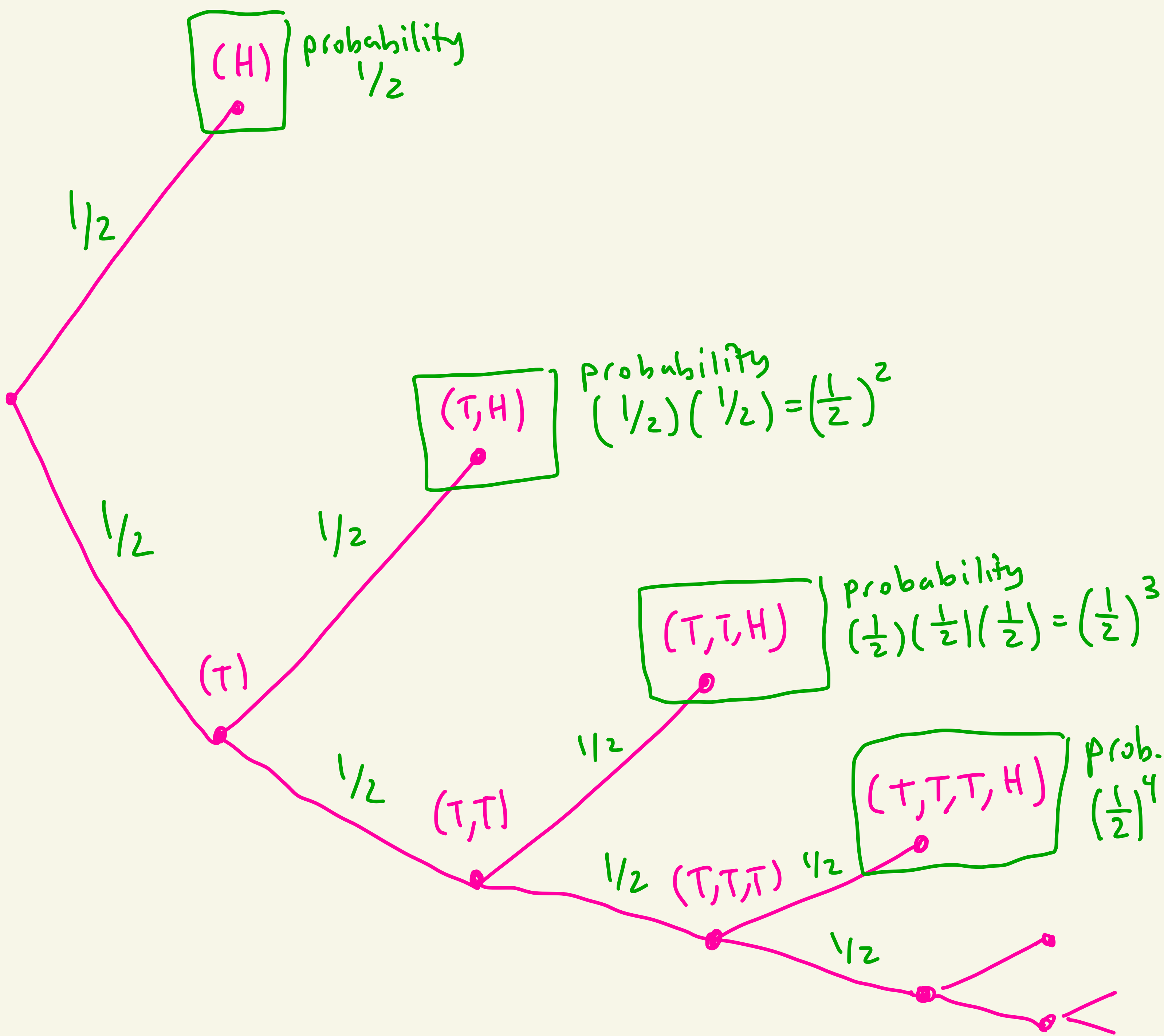
$$E[X] = (-\$1)\left(\frac{125}{216}\right) + (\$1)\left(\frac{75}{216}\right) + (\$2)\left(\frac{15}{216}\right) + (\$3)\left(\frac{1}{216}\right)$$

$$= -\$\frac{17}{216} \approx -\$0.0787$$

(7) From previous HW, the probability space is

$$S = \{(H), (T,H), (T,T,H), (T,T,T,H), (T,T,T,T,H), (T,T,T,T,T,H), \ldots\}$$

The probability tree looks like this:



Let

$E = \{(T,T,T,H), (T,T,T,T,H), (T,T,T,T,T,H), \dots\}$

We want $P(E)$.

You could calculate this in two ways.

Method 1

$$P(E) = P(\{(T,T,T,H)\}) + P(\{(T,T,T,T,H)\}) + \dots$$

$$= \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^6 + \left(\tfrac{1}{2}\right)^7 + \dots$$

$$= \left(\tfrac{1}{2}\right)^4 \left[1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \dots\right]$$

$$= \frac{1}{16}\left[\frac{1}{1-1/2}\right] = \frac{1}{16}\left[2\right] = \frac{1}{8}$$

$$\boxed{1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{if } -1 < x < 1}$$

Method 2

$$P(E) = 1 - P(\overline{E}) = 1 - P(\{(H), (T,H), (T,T,H)\})$$

$$= 1 - \left[\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3\right]$$

$$= 1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8} = \frac{8-4-2-1}{8}$$

$$= 1/8$$

Thus, $P(\overline{E}) = 1 - P(E) = 1 - \frac{1}{8} = \frac{7}{8}$

Let X be the amount won or lost.

Then

$$E[X] = (\$5)P(E) + (-\$1)P(\overline{E})$$

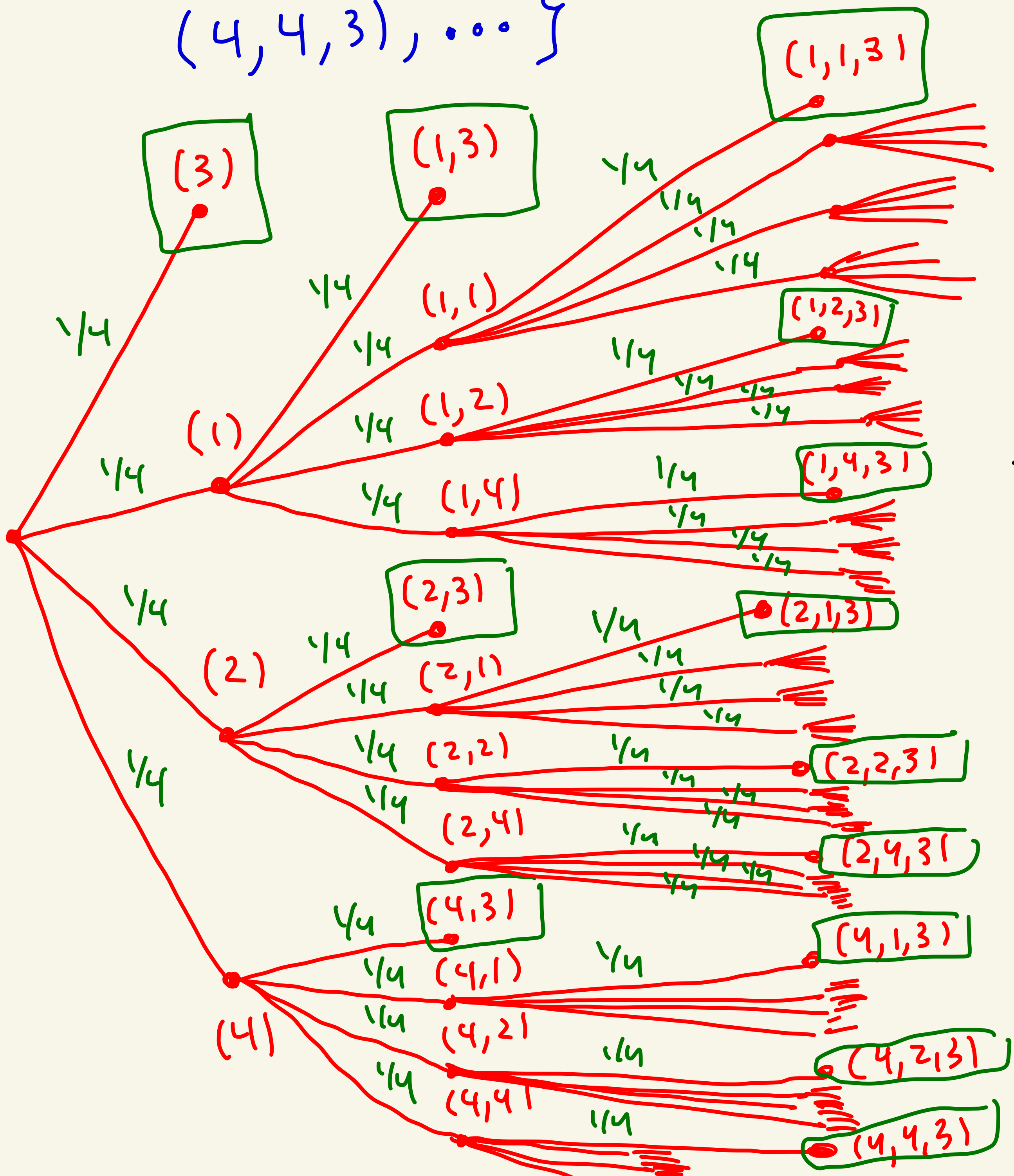
$$= (\$5)\left(\frac{1}{8}\right) + (-\$1)\left(\frac{7}{8}\right)$$

$$= -\$\frac{2}{8} = \boxed{-\$0.25}$$

The expected value is negative so in the long run if you did this bet many times you would expect to lose \$0.25 per bet.

8 (a)

$S = \{ (3), (1,3), (2,3), (4,3),$
$(1,1,3), (1,2,3), (1,4,3), (2,1,3),$
$(2,2,3), (2,4,3), (4,1,3), (4,2,3),$
$(4,4,3), \ldots \}$



The probability tree is on the left. The boxed elements are S. Each branch is probability 1/4. You multiply the probabilities see next page ↓

We have

$$P(\{(3)\}) = \frac{1}{4}$$

$$P(\{(1,3)\}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$$

$$P(\{(2,3)\}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$$

$$P(\{(4,3)\}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$$

$$P(\{(1,1,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(1,2,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(1,4,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(2,1,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(2,2,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(2,4,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

and so on ...

We check that P is a probability function by showing that the sum of P over S is 1. We have

$$\sum_{w \in S} P(\{w\}) = P(\{(3)\}) + P(\{(1,3)\}) + P(\{(2,3)\})$$
$$+ P(\{(4,3)\}) + P(\{(1,1,3)\})$$
$$+ P(\{(1,2,3)\}) + P(\{(1,4,3)\})$$
$$+ P(\{(2,1,3)\}) + P(\{(2,2,3)\})$$
$$+ P(\{(2,4,3)\}) + P(\{(4,1,3)\})$$
$$+ P(\{(4,2,3)\}) + P(\{(4,4,3)\})$$
$$+ \cdots$$
$$= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^3}$$
$$+ \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3}$$
$$+ \frac{1}{4^3} + \cdots$$
$$= \frac{1}{4} + 3 \cdot \frac{1}{4^2} + 3^2 \cdot \frac{1}{4^3} + 3^3 \cdot \frac{1}{4^4} + \cdots$$
$$= \frac{1}{4}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots\right]$$

$$= \frac{1}{4}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \dots\right]$$

$$= \frac{1}{4}\cdot\left[\frac{1}{1-3/4}\right] = \frac{1}{4}\cdot\left[\frac{1}{1/4}\right] = 1$$

$$\boxed{1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{if } -1 < x < 1}$$

Thus, P is a probability function on the space S.

(b) $A = \{(1,1,3), (1,2,3), (1,4,3), (2,1,3), (2,2,3), (2,4,3), (4,1,3), (4,2,3), (4,4,3)\}$

$$P(A) = \underbrace{\frac{1}{4^3} + \frac{1}{4^3} + \dots + \frac{1}{4^3}}_{9 \text{ elements}} = 9\cdot\frac{1}{4^3} = \frac{9}{64} \approx 0.1406\ldots$$

$$\approx 14\%$$

(c) $B = \{(3), (1,3), (2,3), (4,3), (1,1,3), (1,2,3), (1,4,3), (2,1,3), (2,2,3), (2,4,3), (4,1,3), (4,2,3), (4,4,3)\}$

$$P(B) = \frac{1}{4} + 3 \cdot \frac{1}{4^2} + 9 \cdot \frac{1}{4^3} = \frac{16+12+9}{64}$$

$$= \frac{37}{64} \approx 0.578125\ldots \approx 57.8\%$$

(d) X = amount won or lost

$$E[X] = (\$5)\underbrace{\left(\frac{37}{64}\right)}_{\substack{\text{probability} \\ \text{3 is rolled} \\ \text{within first} \\ \text{3 rolls}}} + (-\$6)\underbrace{\left(\frac{27}{64}\right)}_{\substack{\text{probability} \\ \text{3 is rolled after} \\ \text{first 3 rolls}}}$$

$$= \frac{\$185 - \$162}{64} = \$\frac{23}{64} \approx \$0.359$$

If you can play the game many times then you'd expect to win on average \$0.36 per game. So good to play if you can play many times.