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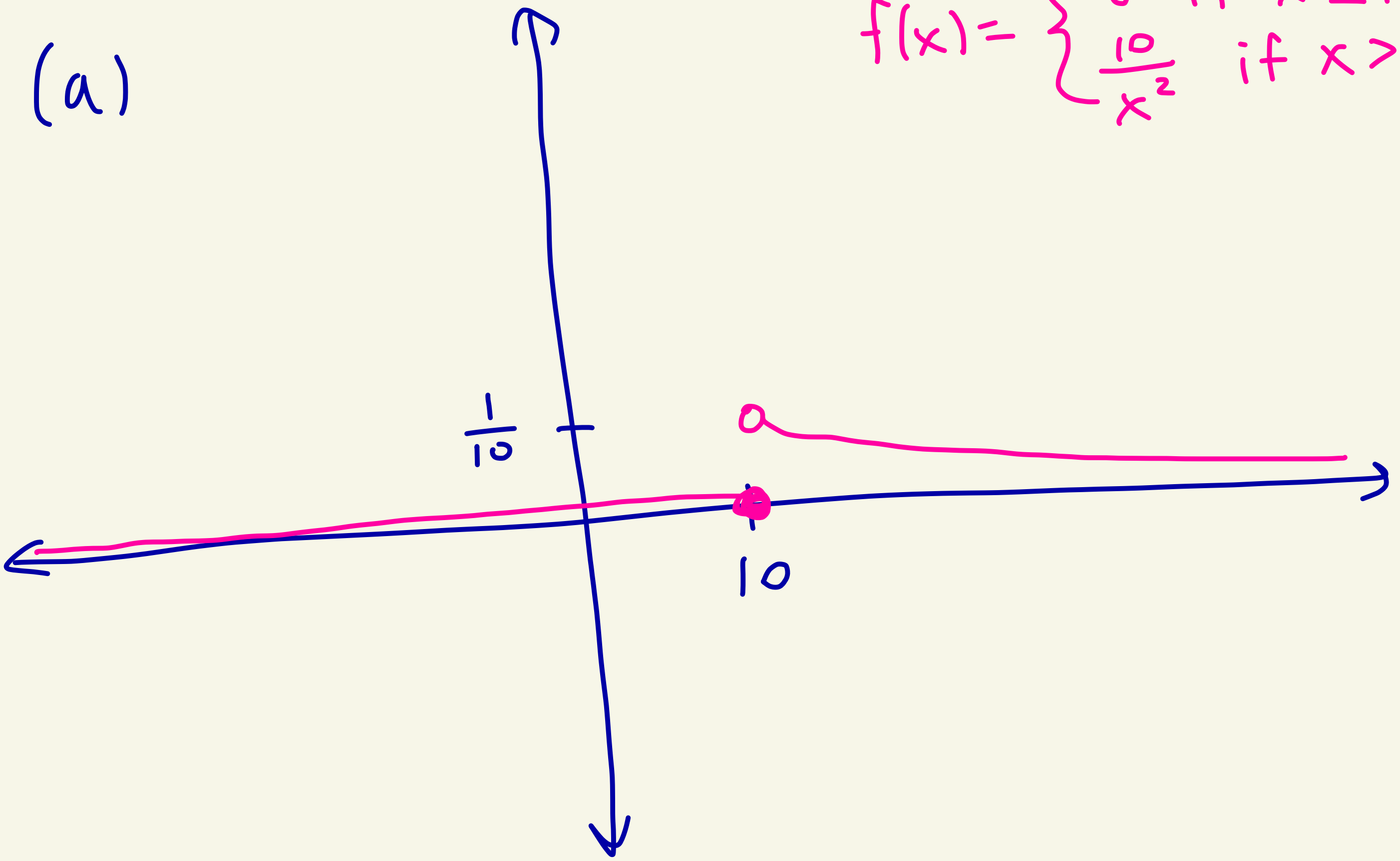
HW 8 Solutions

1 (a)

$$f(x) = \begin{cases} 0 & \text{if } x \le 10 \\ \frac{10}{x^2} & \text{if } x > 10 \end{cases}$$



1 (b) We need to show that

- $f(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

We see that $f(x) \geq 0$ because $0 \geq 0$ and $\frac{10}{x^2} \geq 0$ for all x.

Note that

$$\int \frac{10}{x^2}\,dx = \int 10x^{-2}\,dx = 10\,\frac{x^{-1}}{-1} + C$$

$$= -\frac{10}{x} + C$$

There is an asymptote at $x=0$ so when we integrate we have to define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{10} 0\,dx + \int_{10}^{\infty} \frac{10}{x^2}\,dx$$

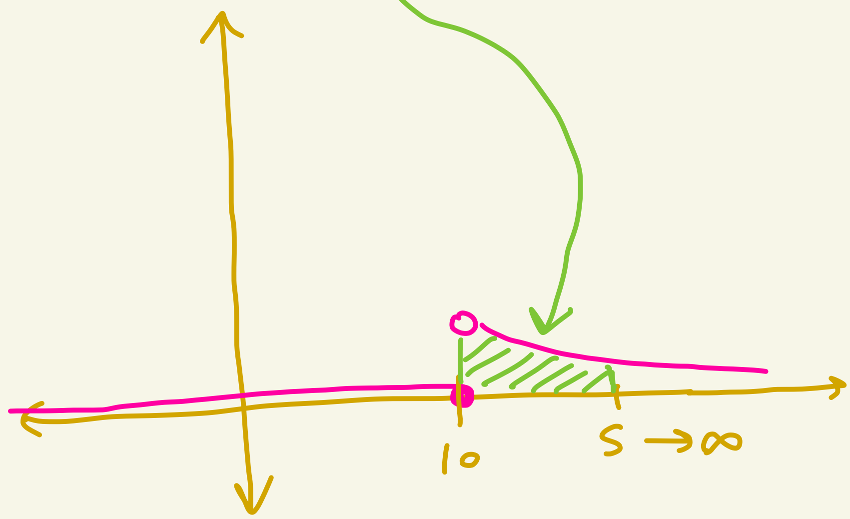
def of f

$$= \int_{10}^{\infty} \frac{10}{x^2}\,dx$$

This is an improper integral.

We have

$$\int_{10}^{\infty} \frac{10}{x^2}\,dx = \lim_{s\to\infty}\int_{10}^{s}\frac{10}{x^2}\,dx =$$



$$= \lim_{s\to\infty}\left(\frac{-10}{x}\right)_{10}^{s} = \lim_{s\to\infty}\left(\frac{-10}{s} - \left(\frac{-10}{10}\right)\right)$$

$$= -\underbrace{\lim_{s\to\infty}\frac{10}{s}}_{0} + \lim_{s\to\infty} 1$$

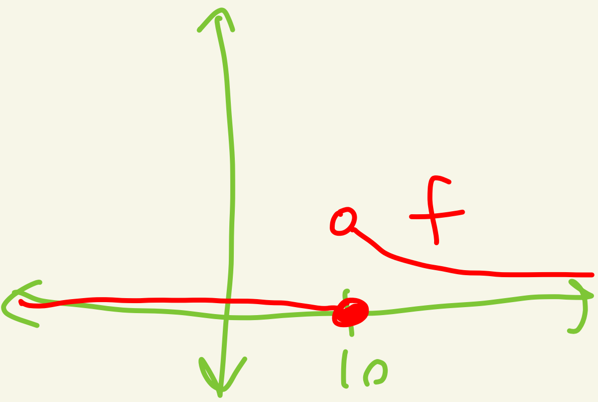
$$= 0+1 = 1$$

We have shown that f is a probability

(1)(c) $P(1 \le X \le 5)$ is defined as

$$P(1 \le X \le 5) = \int_1^5 f(x)\,dx = 0$$

Since $f(x) = 0$ when $1 \le x \le 5$

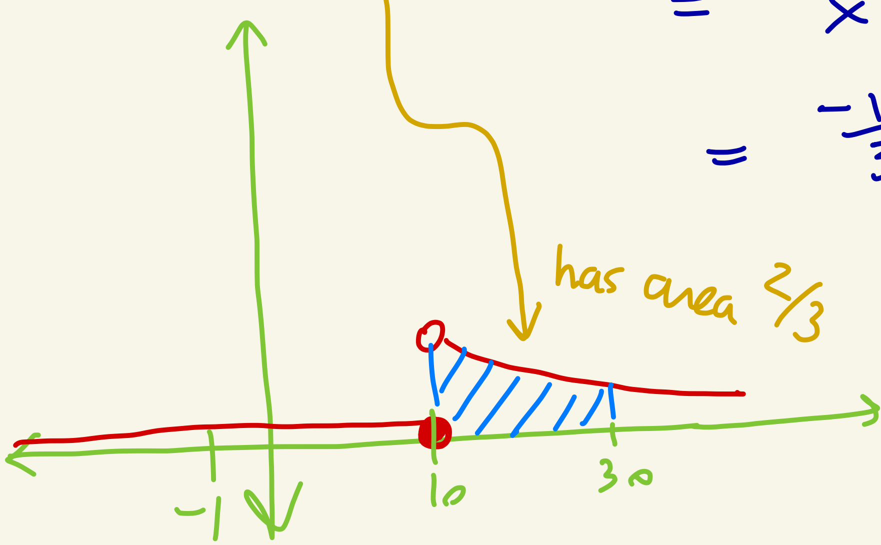


(1)(d) We have that

$$\underbrace{P(-1 \le X \le 30) = \int_{-1}^{30} f(x)\,dx}_{} = \int_{10}^{30} f(x)\,dx$$

$$= \frac{-10}{x}\Big|_{10}^{30} = \frac{-10}{30} - \left(\frac{-10}{10}\right)$$

$$= -\frac{1}{3} + 1 = 1 - \frac{1}{3} = \frac{2}{3}$$



① (e)

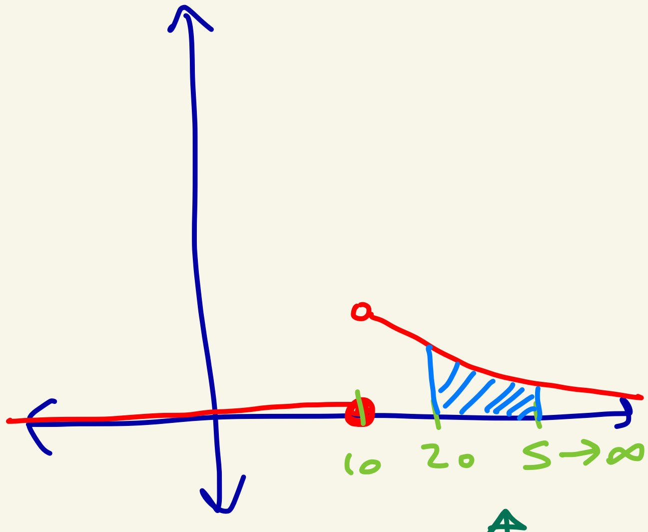
$$P(X>20) = \int_{20}^{\infty} f(x)\,dx = \int_{20}^{\infty} \frac{10}{x^2}\,dx$$

$$= \lim_{s\to\infty} \int_{20}^{s} \frac{10}{x^2}\,dx$$

$$= \lim_{s\to\infty} \left(\frac{-10}{x}\right)_{20}^{s}$$

$$= \lim_{s\to\infty} \left(\frac{-10}{s}\right) - \left(\frac{-10}{20}\right)$$

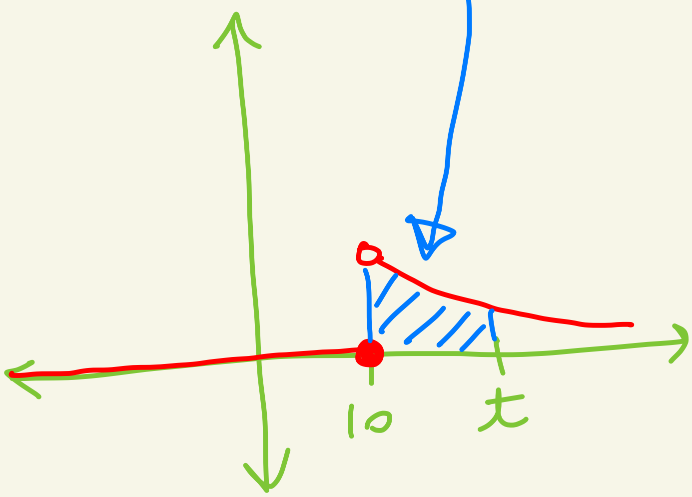
$$= \lim_{s\to\infty} \left(\underbrace{\frac{-10}{s}}_{0} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}$$



has area 1/2

① (f)

$$F(t) = \underbrace{\int_{-\infty}^{t} f(x)\,dx}$$



Note that if $t \le 10$ then

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{t} 0\,dx = 0$$

However if $t > 0$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \underbrace{\int_{-\infty}^{10} 0\,dx}_{0} + \int_{10}^{t} \frac{10}{x^2}\,dx$$

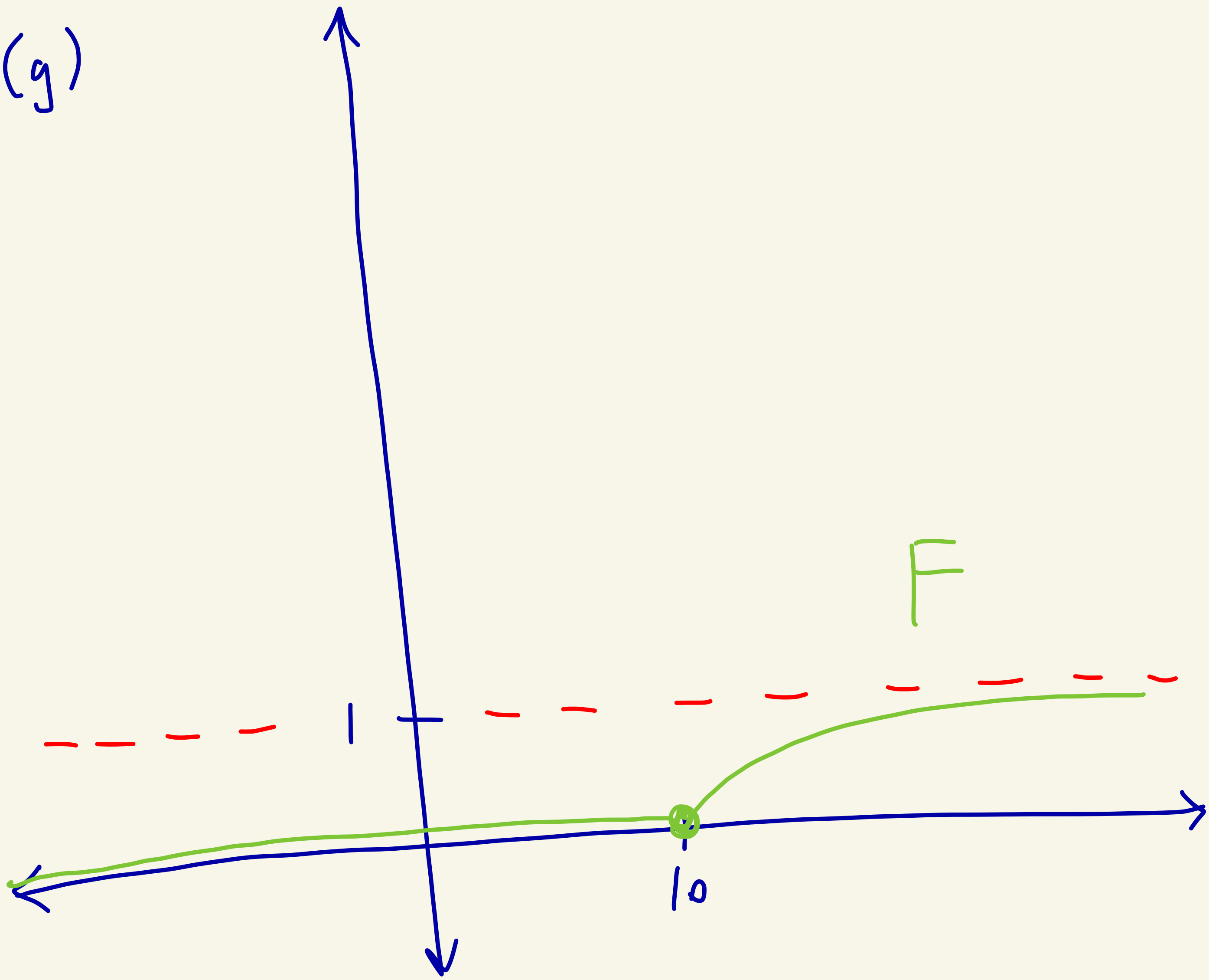
$$= \int_{10}^{t} \frac{10}{x^2}\,dx = \frac{-10}{x}\Big|_{10}^{t} = \frac{-10}{t} - \left(\frac{-10}{10}\right)$$

$$= 1 - \frac{10}{t}$$

Thus,

$$F(t) = \begin{cases} 0 & \text{if } t \le 10 \\ 1 - \frac{10}{t} & \text{if } t > 10 \end{cases}$$

(g)



(h)

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{10} x \cdot 0 \, dx}_{0} + \int_{10}^{\infty} x \cdot \frac{10}{x^2}\, dx$$

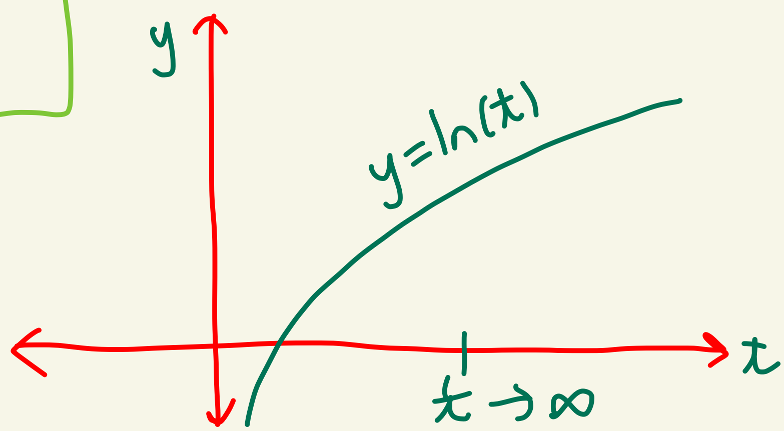
$$= \int_{10}^{\infty} \frac{10}{x}\, dx = \lim_{t\to\infty} \int_{10}^{t} \frac{10}{x}\, dx$$

$$\boxed{\int \frac{1}{x}\, dx = \ln(x) + C}$$

$$= \lim_{t\to\infty} 10 \ln(x) \Big|_{10}^{t}$$

$$= 10 \cdot \lim_{t\to\infty} \left[\ln(t) - \ln(10) \right] = \infty$$

The expected value is infinite.



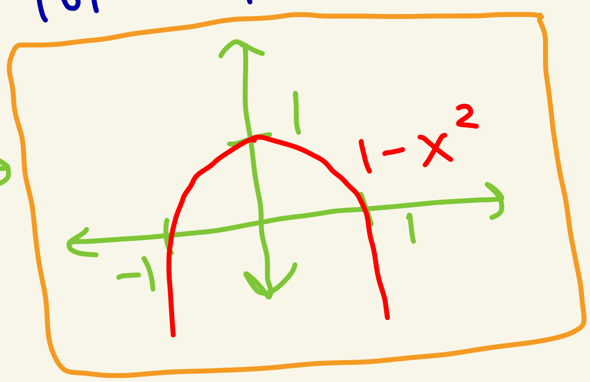
②

(a)

First we need $f(x) \geq 0$ for all x.

So we need $c(1-x^2) \geq 0$ for $-1 \leq x \leq 1$

Note that $1-x^2 \geq 0$ when $-1 \leq x \leq 1$.



Thus we need $c \geq 0$.

We also need $\int_{-\infty}^{\infty} f(x)dx = 1$.

Since $f(x) = \begin{cases} c(1-x^2) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

we have that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} c(1-x^2)\,dx = \int_{-1}^{1} (c - cx^2)dx$$

$$= cx - c\frac{x^3}{3} \Big|_{-1}^{1} = \left[\left(c - \frac{c}{3}\right) - \left(-c + \frac{c}{3}\right)\right]$$

$$= 2c - \frac{2c}{3} = \frac{6c-2c}{3} = \frac{4c}{3}$$

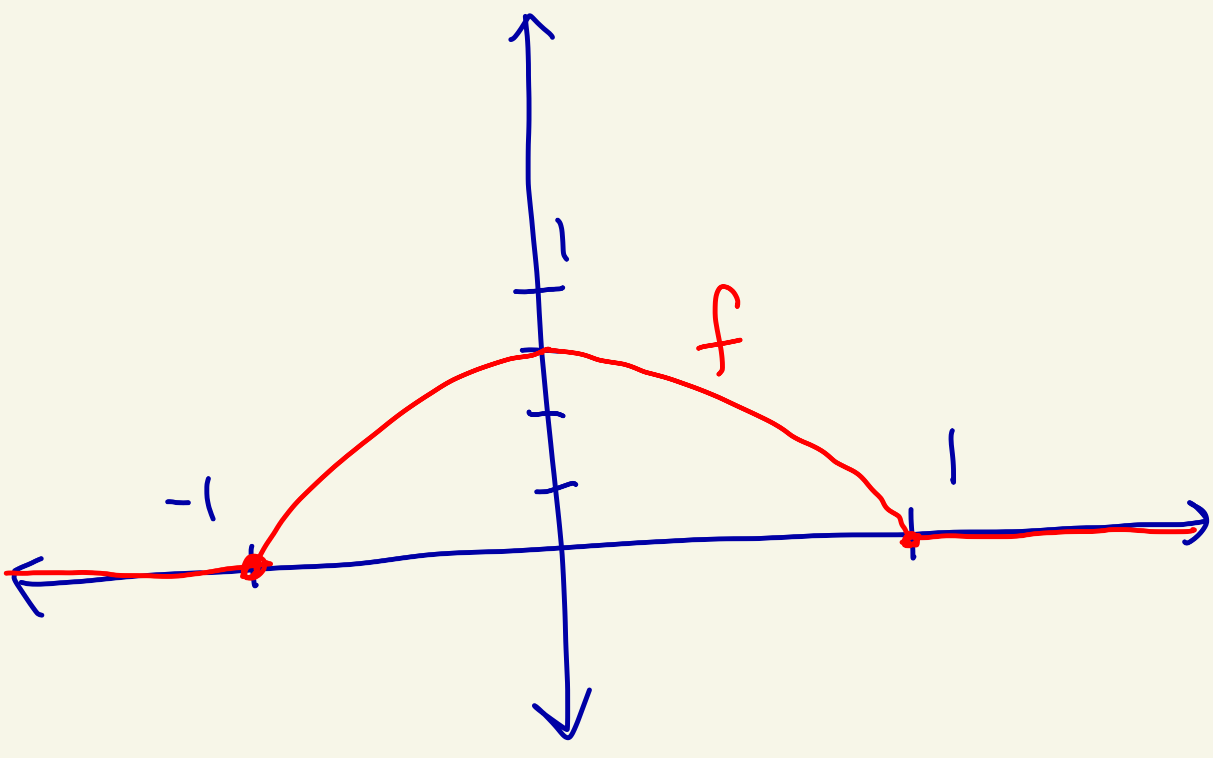
Thus, we need $\frac{4c}{3} = 1$.

Thus, $\boxed{c = \frac{3}{4}}$

So,

$$f(x) = \begin{cases} \frac{3}{4} - \frac{3}{4}x^2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2 (b)



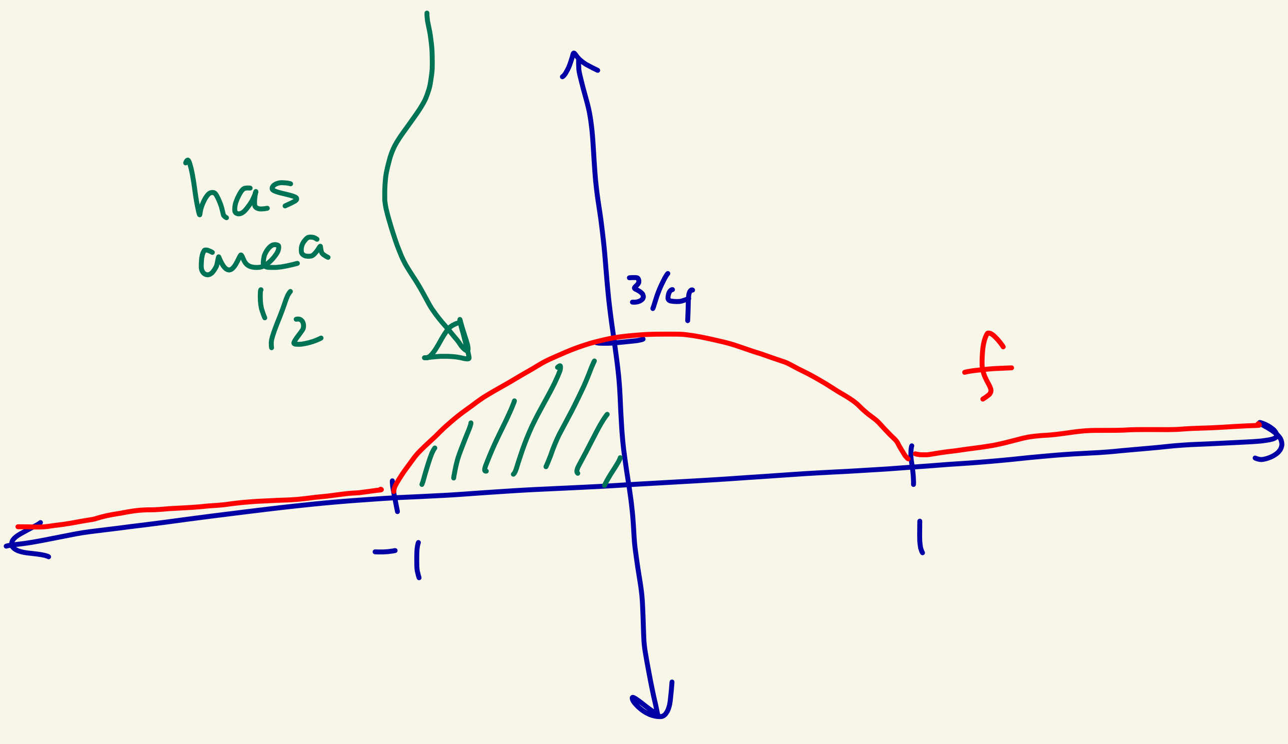
② (c)

$$P(x<0) = \int_{-\infty}^{0} f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{-1} 0\,dx}_{0} + \int_{-1}^{0} \left(\frac{3}{4} - \frac{3}{4}x^2\right) dx$$

$$= \frac{3}{4}x - \frac{3}{4}\frac{x^3}{3} \Big|_{-1}^{0} = (0) - \left(\frac{3}{4}(-1) - \frac{1}{4}(-1)^3\right)$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$



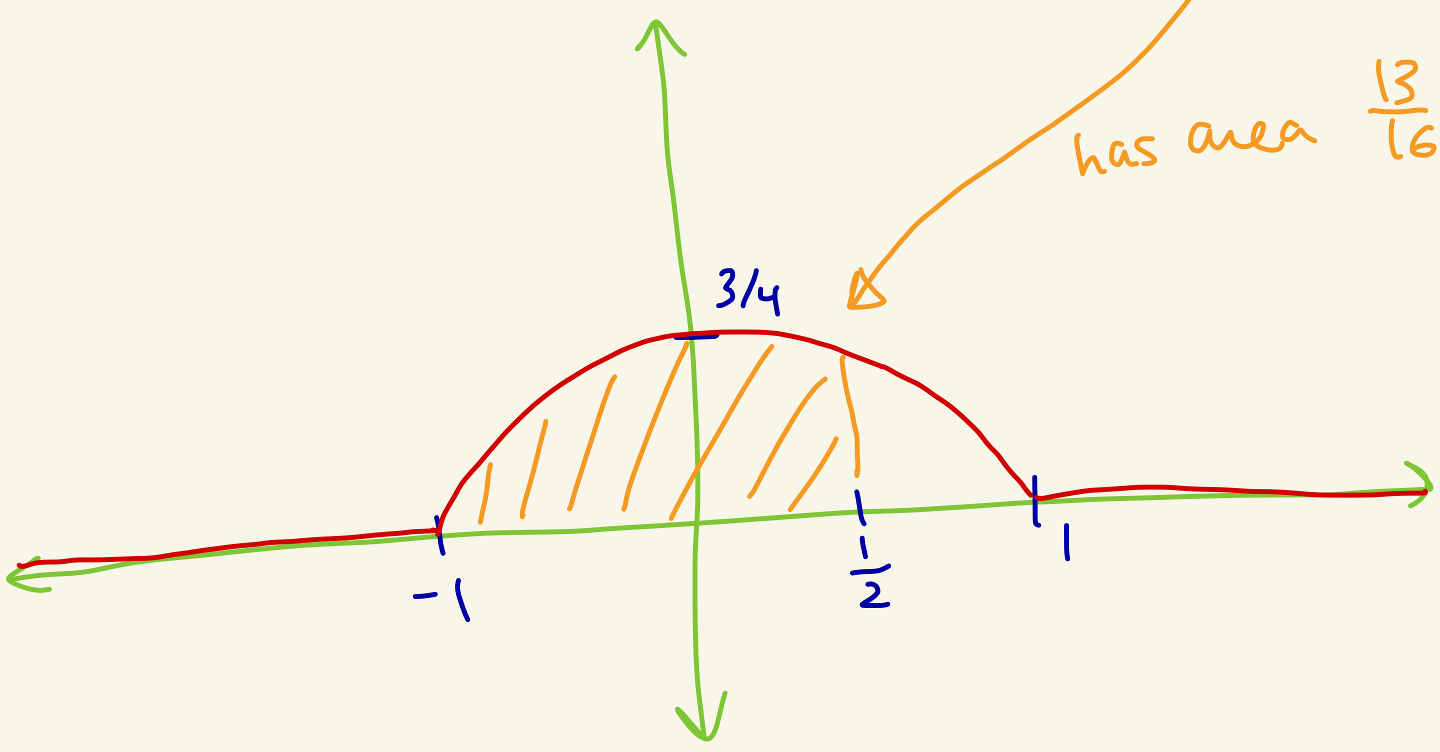
② (d)

$$P(-1 \le X < 1/2) = \int_{-1}^{1/2} f(x)\,dx$$

$$= \int_{-1}^{1/2} \left(\frac{3}{4} - \frac{3}{4}x^2\right) dx$$

$$= \left[\frac{3}{4}x - \frac{1}{4}x^3\right]_{-1}^{1/2} = \left[\frac{3}{4}\cdot\frac{1}{2} - \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2\right] - \left[\frac{3}{4}(-1) - \frac{1}{4}(-1)^3\right]$$

$$= \frac{3}{8} - \frac{1}{16} + \frac{3}{4} - \frac{1}{4} = \frac{6-1+12-4}{16} = \boxed{\frac{13}{16}}$$



② (e)

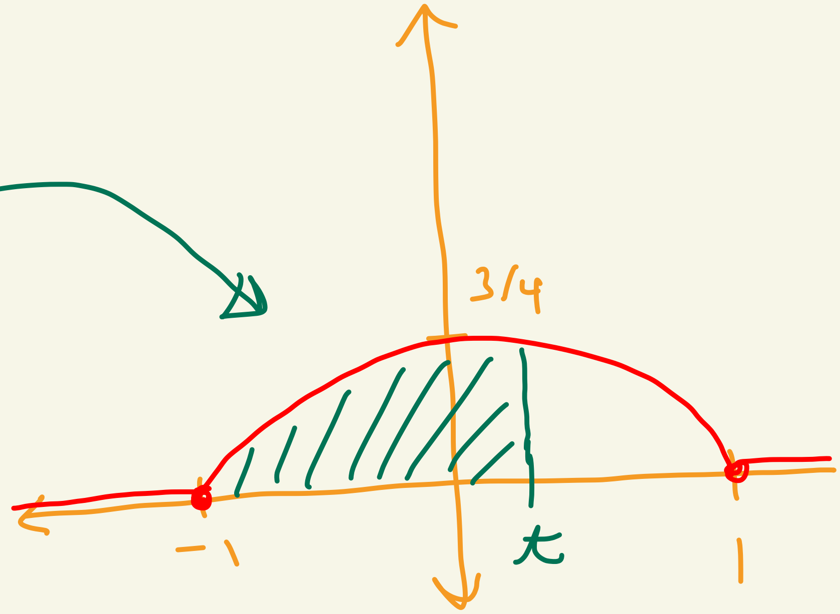
$$P\left(-10 \le X < \tfrac{1}{2}\right) = \int_{-10}^{1/2} f(x)\,dx$$

$$= \underbrace{\int_{-10}^{-1} 0\,dx}_{0} + \int_{-1}^{1/2} f(x)\,dx$$

$$= \int_{-1}^{1/2} f(x)\,dx = \frac{13}{16}$$ from 2d.

② (f)

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$



If $t \leq -1$ then

$$F(t) = \int_{-\infty}^{t} 0\,dx = 0$$

If $-1 \leq t \leq 1$, then

$$F(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-1}^{t} \left(\frac{3}{4} - \frac{3}{4}x^2\right)dx$$

$$= \left[\frac{3}{4}x - \frac{1}{4}x^3\right]_{-1}^{t} = \left[\frac{3}{4}t - \frac{1}{4}t^3\right] - \left[-\frac{3}{4} + \frac{1}{4}\right]$$

$$= \frac{1}{2} + \frac{3}{4}t - \frac{1}{4}t^3$$

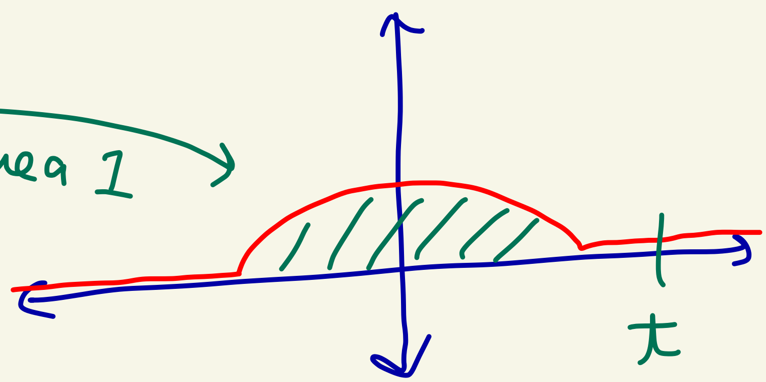
If $1 \le t$, then

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{-1} 0\,dx}_{0} + \underbrace{\int_{-1}^{1} \left(\tfrac{3}{4} - \tfrac{3}{4}x^2\right)dx}_{1} + \underbrace{\int_{1}^{t} 0\,dx}_{0}$$

$$= 1$$

has area 1

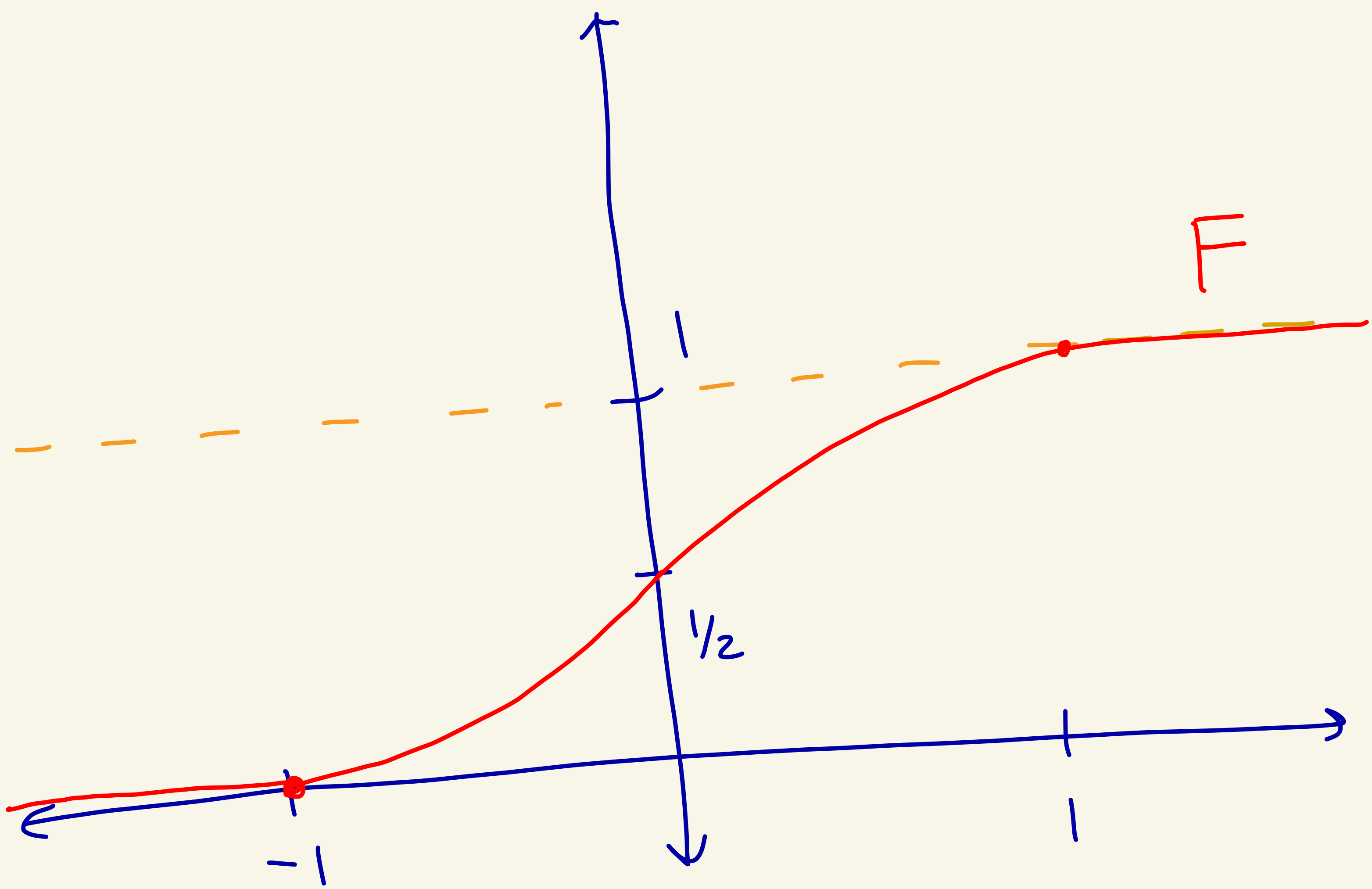


Thus,

$$F(t) = \begin{cases} 0 & \text{if } t \le -1 \\ \frac{1}{2} + \frac{3}{4}t - \frac{1}{4}t^3 & \text{if } -1 \le t \le 1 \\ 1 & \text{if } 1 \le t \end{cases}$$

② (g)

$$F(t) = \begin{cases} 0 & \text{if } t \le -1 \\ \frac{1}{2} + \frac{3}{4}t - \frac{1}{4}t^3 & \text{if } -1 \le t \le 1 \\ 1 & \text{if } 1 \le t \end{cases}$$



2 (h)

$$E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$= \int_{-\infty}^{-1} x \cdot 0 \, dx + \int_{-1}^{1} x\left(\frac{3}{4} - \frac{3}{4}x^2\right) dx + \int_{1}^{\infty} x \cdot 0 \, dx$$

$$= \int_{-1}^{1} \left(\frac{3}{4}x - \frac{3}{4}x^3\right) dx = \frac{3}{4}\frac{x^2}{2} - \frac{3}{4}\frac{x^4}{4} \Bigg|_{-1}^{1}$$

$$= \left[\frac{3}{8} - \frac{3}{16}\right] - \left[\frac{3}{8} - \frac{3}{16}\right]$$

$$= 0$$

③ (a) Let $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that $\underbrace{\lambda}_{\lambda>0} \underbrace{e^{-\lambda x}}_{e^{-\lambda x}>0} > 0$ for all x.

Thus, $f(x) \geq 0$ for all x.

Also,

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} 0\,dx}_{0} + \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

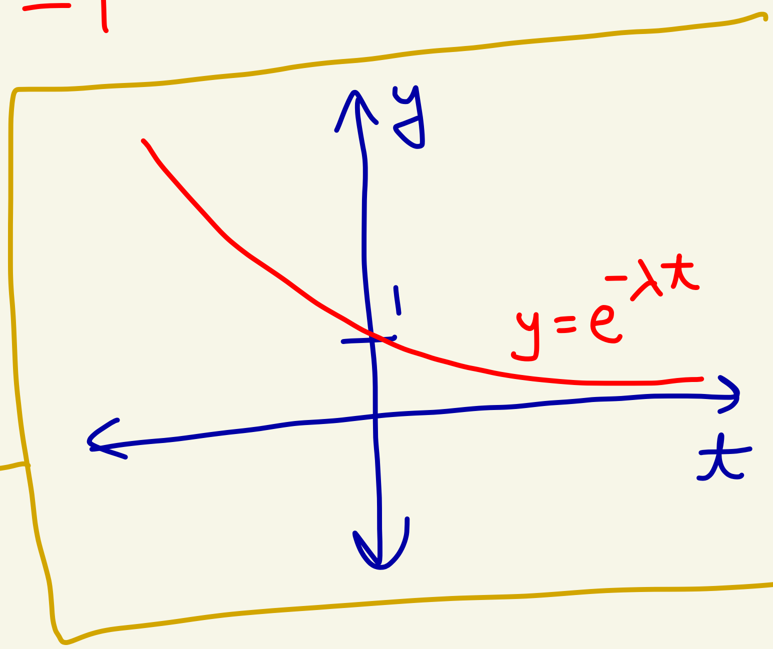
$$= \lim_{t\to\infty} \int_0^t \lambda e^{-\lambda x}\,dx = \lim_{t\to\infty} \left[\lambda\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\right]_0^t$$

$$= \downarrow$$

$$= \lim_{t\to\infty} \left[-e^{-\lambda x} \right]_0^t$$

$$= \lim_{t\to\infty} \left[-e^{-\lambda t} - \underbrace{\left(-e^{-\lambda(0)}\right)}_{-1} \right]$$

$$= 1 - \lim_{t\to\infty} e^{-\lambda t}$$



$$= 1 - 0$$

$$= 1$$

Thus, f is a probability density function.

③ (b)

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \underbrace{\int_{-\infty}^{0} x \cdot 0\,dx}_{0} + \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\,dx$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Note that

$$\int x e^{-\lambda x}\,dx = -\frac{1}{\lambda} x e^{-\lambda x} - \int \left(-\frac{1}{\lambda} e^{-\lambda x}\,dx\right)$$

$$u = x \qquad du = dx$$
$$dv = e^{-\lambda x}\,dx \qquad v = -\frac{1}{\lambda} e^{-\lambda x}$$

$$= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda} \int e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} x e^{-\lambda x} + \frac{1}{\lambda}\left(-\frac{1}{\lambda} e^{-\lambda x}\right) + C$$

$$= -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} + C$$

Thus,

$$E[X] = \lambda \int_0^\infty x e^{-\lambda x}\, dx$$

$$= \lambda \left[\lim_{t \to \infty} \int_0^t x e^{-\lambda x}\, dx \right]$$

$$= \lambda \cdot \lim_{t \to \infty} \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^t$$

$$= \lambda \cdot \lim_{t \to \infty} \left[\left(-\frac{1}{\lambda} t e^{-\lambda t} - \frac{1}{\lambda^2} e^{-\lambda t} \right) - \left(0 - \underbrace{\frac{1}{\lambda^2} e^{-0}}_{-1/\lambda^2} \right) \right]$$

$$= \lambda \cdot \lim_{t \to \infty} \left[\underbrace{\frac{1}{\lambda^2}}_{\to 1/\lambda^2} - \underbrace{\frac{1}{\lambda} t e^{-\lambda t}}_{\text{What about this one?}} - \underbrace{\frac{1}{\lambda^2} e^{-\lambda t}}_{\to 0} \right]$$

like in 3a

$$= \lambda\left[\frac{1}{\lambda^2}\right] - \lambda \cdot \frac{1}{\lambda} \lim_{t\to\infty} t e^{-\lambda t}$$

$$= \frac{1}{\lambda} - \underbrace{\lim_{t\to\infty} \frac{t}{e^{\lambda t}}}_{\text{“}\frac{\infty}{\infty}\text{” situation}}$$

$$= \frac{1}{\lambda} - \lim_{t\to\infty} \underbrace{\frac{1}{\lambda e^{\lambda t}}}_{\frac{1}{\lambda}e^{-\lambda t} \to 0 \text{ as } t\to\infty} = \frac{1}{\lambda} - 0$$

Use L'Hospital rule from calculus

$$= \frac{1}{\lambda}$$

Thus, $E[X] = \dfrac{1}{\lambda}$

(4) Since f is a probability density function we know that

(i) $f(x) \geqslant 0$ for all x

(b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

We can use these two equations to find a & b:

$$\boxed{\begin{gathered}\int_{-\infty}^{\infty} f(x)\,dx = 1 \\ E[X] = \frac{3}{5}\end{gathered}}$$

First equation:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 (a+bx^2)\,dx$$

$f(x)=0$ when $x<0$ or $x>1$

$$= \left(ax + \frac{b}{3}x^3\right)\Big|_0^1 = a + \frac{b}{3}$$

Thus $\boxed{a + \frac{1}{3}b = 1}$

Second equation:

$$\frac{3}{5} = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x(a+bx^2)\, dx = \int_0^1 (ax+bx^3)\, dx$$

f(x) = 0 when x < 0 or x > 1

$$= \left(a\frac{x^2}{2} + b\frac{x^4}{4}\right)\Bigg|_0^1$$

$$= \frac{a}{2} + \frac{b}{4}$$

Thus, $\boxed{\frac{1}{2}a + \frac{1}{4}b = \frac{3}{5}}$

So we have the following linear system:

$$\boxed{\begin{aligned} a + \frac{1}{3}b &= 1 \quad \text{Eqn 1} \\ \frac{1}{2}a + \frac{1}{4}b &= \frac{3}{5} \quad \text{Eqn 2} \end{aligned}}$$

Multiply eqn 2 by -2 and add

$$\begin{array}{l} a + \frac{1}{3}b = 1 \quad \leftarrow \text{Eqn 1} \\ + \left(-a - \frac{1}{2}b = -\frac{6}{5}\right) \quad \leftarrow (2 * \text{Eqn 2}) \\ \hline -\frac{1}{6}b = -\frac{1}{5} \end{array}$$

So, $\boxed{b = \frac{6}{5}}$

Plug this into Eqn 1 to get that

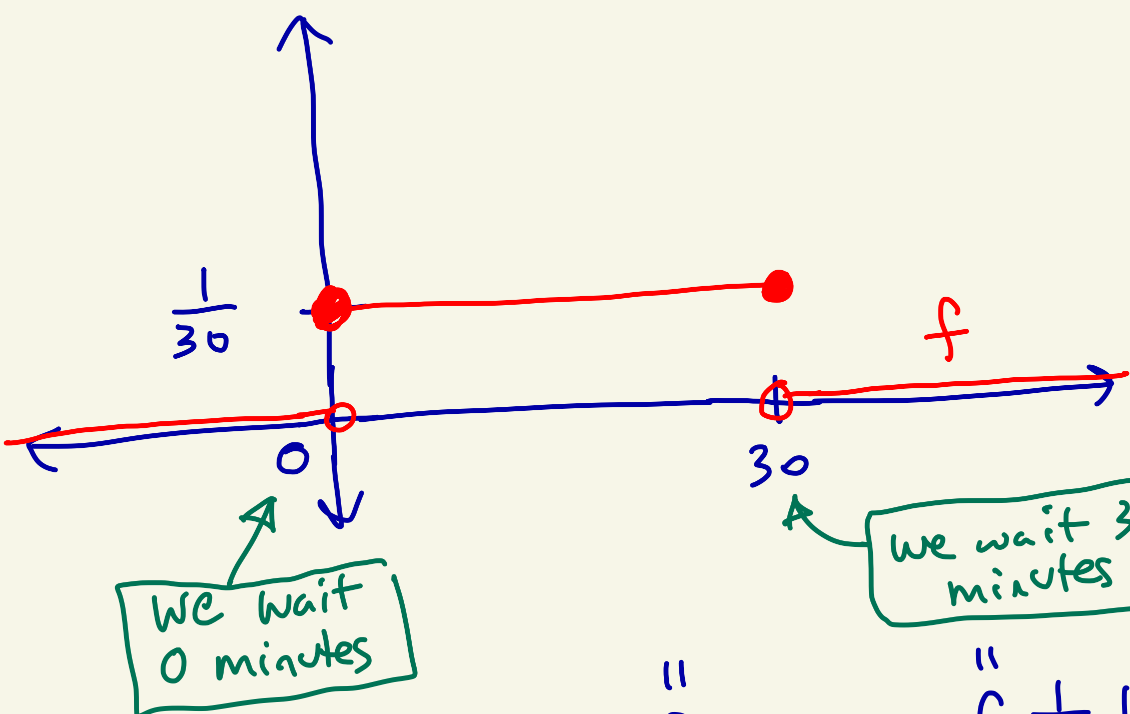
$$a = 1 - \frac{1}{3}b = 1 - \frac{1}{3}\left(\frac{6}{5}\right) = 1 - \frac{6}{15} = \frac{15-6}{15}$$

$$= \frac{9}{15} = \boxed{\frac{3}{5}}$$

Thus, $\boxed{\begin{array}{l} a = \frac{3}{5} \\ b = \frac{6}{5} \end{array}}$

⑤ The bus arrival time is uniformly distributed over a 30 minute time interval. Let X denote the amount of time in minutes that we wait till the bus arrives. So the values of X are $0 \leq X \leq 30$. The uniform density function f for X is given by the following graph.

$$\boxed{\frac{1}{30-0} = \frac{1}{30}}$$



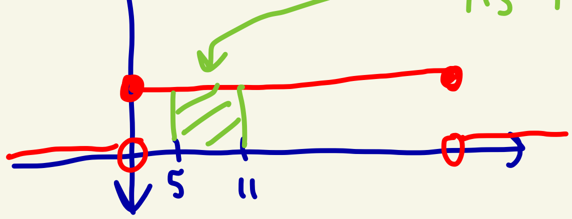
(a) $P(5 \leq X \leq 11) = \int_5^{11} f(x)\,dx = \int_5^{11} \frac{1}{30}\,dx = \frac{1}{30}x \Big|_5^{11}$

$= \frac{1}{30}(11-5) = \frac{6}{30} = \frac{1}{5} = \boxed{0.2 = 20\%}$

wait 5 min

wait 11 min

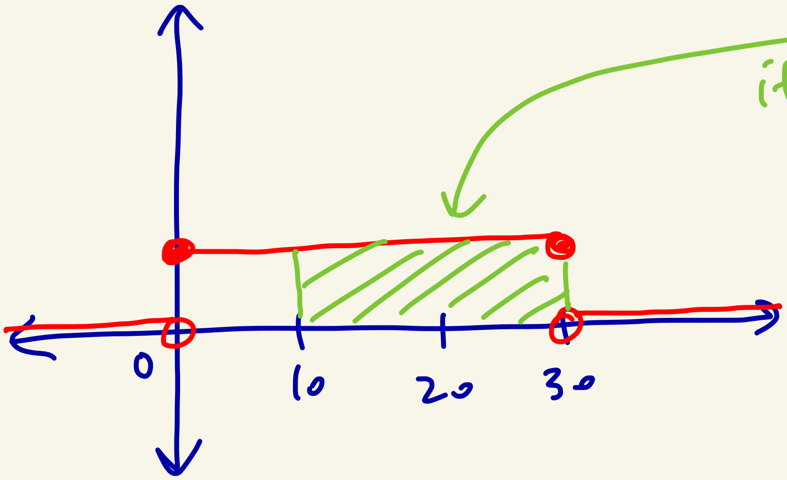
its this area



5b $P(X > 10) = \int_{10}^{\infty} f(x)\,dx = \int_{10}^{30} \frac{1}{30}\,dx$

$= \frac{1}{30}(30-10) = \frac{20}{30} = \boxed{\frac{2}{3} = 0.6\overline{6} \approx 66\%}$

its this area



(6) Let X be the time in hours that it takes to repair a machine and

$$f(x) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

be its probability density function.

(a) $P(0 \leq X \leq 1) = \int_0^1 \frac{1}{2}e^{-\frac{1}{2}x}\,dx$

$$= \frac{1}{2}\left(\frac{1}{(-\frac{1}{2})}e^{-\frac{1}{2}x}\right)\Big|_0^1 = -e^{-\frac{1}{2}x}\Big|_0^1$$

$$= -e^{-1} - (-e^0) = \boxed{1 - \frac{1}{e}} \approx \boxed{0.632\ldots}$$

$$\approx \boxed{63.2\%}$$

(b) $P(X > 2) = \int_2^\infty \frac{1}{2}e^{-1/2x}\,dx = \lim_{t \to \infty} \int_2^t \frac{1}{2}e^{-\frac{1}{2}x}\,dx$

$$= \lim_{t \to \infty}\left(-e^{-1/2x}\Big|_2^t\right) = \lim_{t \to \infty}\left[-e^{-\frac{1}{2}t} - \left(-e^{-\frac{1}{2}(2)}\right)\right]$$

$$= \lim_{t\to\infty}\left[\frac{1}{e} - \underbrace{\frac{1}{e^{\frac{1}{2}t}}}_{0}\right] = \frac{1}{e} - 0 = \boxed{\frac{1}{e}}$$

$$\approx \boxed{0.367879}$$

$$\approx \boxed{36.8\%}$$