Math 5401 - Test 2

Name: Solutions

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1. [12 points - 4,4,4] Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $H = \langle(\bar{0}, \bar{3})\rangle = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{3})\}$.

(a) Compute the elements of the group G/H. How many elements are there?

(b) Find the order of the element $(\bar{0}, \bar{1}) + H$ in G/H.

(c) Is G/H cyclic? Why or why not?

(a)

$$\left.\begin{array}{l}
(\bar{0},\bar{0})+H = \{(\bar{0},\bar{0}),(\bar{0},\bar{3})\} \\
(\bar{1},\bar{0})+H = \{(\bar{1},\bar{0}),(\bar{1},\bar{3})\} \\
(\bar{0},\bar{1})+H = \{(\bar{0},\bar{1}),(\bar{0},\bar{4})\} \\
(\bar{1},\bar{1})+H = \{(\bar{1},\bar{1}),(\bar{1},\bar{4})\} \\
(\bar{0},\bar{2})+H = \{(\bar{0},\bar{2}),(\bar{0},\bar{5})\} \\
(\bar{1},\bar{2})+H = \{(\bar{1},\bar{2}),(\bar{1},\bar{5})\}
\end{array}\right\}$$

G/H has 6 elements.

(b) $(\bar{0},\bar{1})+H \neq (\bar{0},\bar{0})+H$

$[(\bar{0},\bar{1})+H]+[(\bar{0},\bar{1})+H] = (\bar{0},\bar{2})+H \neq (\bar{0},\bar{0})+H$

$[(\bar{0},\bar{1})+H]+[(\bar{0},\bar{1})+H]+[(\bar{0},\bar{1})+H] = (\bar{0},\bar{3})+H = (\bar{0},\bar{0})+H$

$(\bar{0},\bar{1})+H$ has order 3

(c)

element	order
$(\bar{0},\bar{0})+H$	1
$(\bar{1},\bar{0})+H$	2
$(\bar{0},\bar{1})+H$	3
$(\bar{1},\bar{1})+H$	6
$(\bar{0},\bar{2})+H$	3
$(\bar{1},\bar{2})+H$	6

G/H is cyclic

It has 2 generators

2. [10 points] Consider the group $G = \mathbb{Z} \times \mathbb{Z}$ and the subgroup $H = \langle (4,1) \rangle$. Use the first isomorphism theorem to show that G/H is isomorphic to a familiar group.

See HW 3.3 B for another example like this one

$G/H \cong \mathbb{Z}$

Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by

$\varphi(x,y) = x - 4y$.

φ is a homomorphism

Let $(a,b), (x,y) \in \mathbb{Z} \times \mathbb{Z}$. Then

$\varphi((a,b)+(x,y)) = \varphi(a+x, b+y) = (a+x) - 4(b+y)$

$= a - 4b + x - 4y = \varphi(a,b) + \varphi(x,y)$.

φ is onto $\mathbb{Z}$

Let $z \in \mathbb{Z}$. Then $(z,0) \in \mathbb{Z} \times \mathbb{Z}$ and

$\varphi(z,0) = z - 4 \cdot 0 = z$.

$\ker \varphi = H$

If $(x,y) \in \ker \varphi$ then $\varphi(x,y) = 0$ and so $x - 4y = 0$. Thus, $x = 4y$. So,

$(x,y) = (4y, y) = y(4,1) \in \langle (4,1) \rangle$.

Conversely, if $(x,y) \in \langle (4,1) \rangle$ then $(x,y) = k(4,1)$ for some $k \in \mathbb{Z}$. So, $\varphi(x,y) = \varphi(4k, k) = 4k - 4 \cdot k = 0$.

So, $(x,y) \in \ker(\varphi)$.

OR → (next page)

2. [10 points] Consider the group $G = \mathbb{Z} \times \mathbb{Z}$ and the subgroup $H = \langle (4, 1) \rangle$. Use the first isomorphism theorem to show that G/H is isomorphic to a familiar group.

Method 2 to show $\ker(\varphi) = H$.

$$\ker(\varphi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid \varphi(x,y) = 0\}$$
$$= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x - 4y = 0.\}$$
$$= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x = 4y\}$$
$$= \{(4y, y) \mid y \in \mathbb{Z}\}$$

$$= \{y(4,1) \mid y \in \mathbb{Z}\}$$
$$= \langle (4,1) \rangle = H$$

3. [15 points - 5 each] True or False. If True, give a proof. If False, give an example showing it is false.

(a) Let G be abelian and $H \leq G$. Then G/H is abelian.

True.

Let $aH, bH \in G/H$ where $a, b \in G$. Then $ab = ba$ since G is abelian. So,

$(aH)(bH) = abH = baH = (bH)(aH)$.

So, G/H is abelian.

HW 3.1 3(a)

(b) There does not exist a non-abelian group G with a normal subgroup H with G/H abelian.

False,

Let $G = D_6 = \{1, r, r^2, s, sr, sr^2\}$

and $H = \langle r \rangle = \{1, r, r^2\}$.

left cosets	right cosets
$H = \{1, r, r^2\}$	$H = \{1, r, r^2\}$
$sH = \{s, sr, sr^2\}$	$Hs = \{s, rs, r^2s\} = \{s, sr^2, sr^2\}$

$H = H$

$sH = Hs$

So, $H \trianglelefteq D_6$.

And $D_6/H = \{H, sH\} \cong \mathbb{Z}_2$

so, D_6/H is abelian.

(3 continued...) True or False. If True, give a proof. If False, give an example showing it is false.

(c) If G is a group of size 10 and H is a subgroup of G with $H \neq G$, then H is cyclic.

True.

Suppose $H \leq G$ with $|G| = 10$, and $H \neq G$

By Lagrange, $|H|$ divides $|G|$.

Since $|H| \neq |G|$ we have $|H| = 1$, 2, or 5.

If $|H| = 1$, then $H = \{1\} = \langle 1 \rangle$ is cyclic.

If $|H| = 2$, then since 2 is prime H is cyclic.

If $|H| = 5$, then since 5 is prime H is cyclic.

4. [10 points] Pick <u>ONE</u> of the following.

A) Let G be a group. (a) Prove that if $x \in G$ then $|x|$ divides $|G|$. (b) Prove that if $|G| = p$ where p is prime, then G is cyclic.

B) Let G and H be groups and $\phi : G \to H$ be a homomorphism. (a) Prove that if $E \leq H$, then $\phi^{-1}(E) \leq G$. (b) Prove that if $E \trianglelefteq H$, then $\phi^{-1}(E) \trianglelefteq G$.

(A) In class. See test 2 study guide.

(B) HW 3.1 #1(a,b).

5. [10 points] Let $G = H \times K$ where H and K are groups. Let $K_1 = \{(1_H, k) \mid k \in K\}$ where 1_H is the identity element of H.

(a) Prove that K_1 is a normal subgroup of G.

(b) Prove that G/K_1 is isomorphic to H.

(a) $\underline{K_1 \leq G}$

- $(1_H, 1_K) \in K_1$ by def of K_1
- Let $a, b \in K_1$ where $a = (1_H, k_1)$ and $b = (1_H, k_2)$, and $k_1, k_2 \in K$. Then,

$$ab^{-1} = (1_H, k_1)(1_H, k_2^{-1}) = (1_H, k_1 k_2^{-1}) \in K_1$$

since $k_1 k_2^{-1} \in K$ because K is a group.

$\underline{K_1 \trianglelefteq G}$

Let $g \in G$ and $k \in K_1$. Then $g = (a, b)$ and $k = (1_H, k_1)$ where $a \in H$, $b, k_1 \in K$.

So, $gkg^{-1} = (a, b)(1_H, k_1)(a, b^{-1}) = (a 1_H a^{-1}, b k_1 b^{-1})$

$= (1_H, b k_1 b^{-1}) \in K_1$ since $b k_1 b^{-1} \in K$.

(b) Define $\varphi : G \to H$ by $\varphi(h, k) = h$.

$\underline{\varphi \text{ is a homomorphism}}$

Let $(h_1, k_1), (h_2, k_2) \in G$ where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Then $\varphi((h_1, k_1)(h_2, k_2)) = \varphi(h_1 h_2, k_1 k_2)$

$= h_1 h_2 = \varphi(h_1, k_1)\varphi(h_2, k_2)$.

5. [10 points] Let $G = H \times K$ where H and K are groups. Let $K_1 = \{(1_H, k) \mid k \in K\}$ where 1_H is the identity element of H.

(a) Prove that K_1 is a normal subgroup of G.

(b) Prove that G/K_1 is isomorphic to H.

continued...

φ is onto

Let $h \in H$. Then $(h, 1_K) \in G$ and $\varphi(h, 1_K) = h$.

$\ker(\varphi) = K_1$

Let $(a,b) \in \ker(\varphi)$ where $a \in H$ and $b \in K$.
Then $\varphi(a,b) = 1_H$. So, $a = 1_H$.
Thus, $(a,b) = (1_H, b) \in K_1$.
Conversely, given $(1_H, k) \in K_1$ where $k \in K$ we have $\varphi(1_H, k) = 1_H$. So, $(1_H, k) \in \ker(\varphi)$.