7.3 #5

Suppose that $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is a ring hom.

Let $m = \varphi(1,0)$ and $n = \varphi(0,1)$.

Then $m^2 = \varphi(1,0)\varphi(1,0) = \varphi(1 \cdot 1, 0 \cdot 0)$
$= \varphi(1,0) = m$.

Since $m^2 = m$ and $m \in \mathbb{Z}$ we have $m = 0$ or $m = 1$.

Similarly, $n^2 = n$ and so $n = 0$ or $n = 1$.

Consider $(a,b) \in \mathbb{Z}$.

If $a > 1, b > 1$, then $\varphi(a,b) = \varphi(a,0) + \varphi(0,b) =$

$$= \underbrace{\varphi(1,0) + \dots + \varphi(1,0)}_{a \text{ times}} + \underbrace{\varphi(0,1) + \dots + \varphi(0,1)}_{b \text{ times}}$$

$$= a\varphi(1,0) + b\varphi(0,1) = am + bn.$$

If $a < 1, b < 1$, then $\varphi(a,b) = \varphi(a,0) + \varphi(0,b)$

$$= \underbrace{\varphi(-1,0) + \dots + \varphi(-1,0)}_{-a \text{ times}} + \underbrace{\varphi(0,-1) + \dots + \varphi(0,-1)}_{-b \text{ times}}$$

$$= -a\left[\varphi(-1,0)\right] - b\varphi(0,-1)$$

$$= -a\left[-\varphi(1,0)\right] - b\left[-\varphi(0,1)\right] = am + bn.$$

You can show that for all cases that $\varphi(a,b) = am + bn$.

So, given that $m = 0, 1$, and $n = 0, 1$ are the only possibilities we get these possibilities for φ:

$\varphi(a,b) = 0$, $\varphi(a,b) = a$, $\varphi(a,b) = b$, $\varphi(a,b) = a+b$

The last case $\varphi(a,b) = a+b$ is not a homomorphism since for example

$$\varphi((2,1)(3,0)) = \varphi(6,0) = 6 + 0 = 6$$

but

$$\varphi(2,1)\varphi(3,0) = (2+1)(3+0) = 9.$$

The other three are ring homomorphisms!

$\varphi(a,b) = 0 \quad \forall (a,b) \in \mathbb{Z}\times\mathbb{Z}$

$\varphi(a,b) = a$

$\varphi(a,b) = b$

You can verify this

Ex: Let $\varphi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ be $\varphi(a,b) = a$

$$\varphi((a,b)+(x,y)) = \varphi(a+x, b+y) = a+x = \varphi(a,b) + \varphi(x,y)$$

and $\varphi((a,b)(x,y)) = \varphi(ax, by) = ax = \varphi(a,b)\,\varphi(x,y)$.