Math 5680

1/25/23

HW 1 Topic – Series

<u>Def:</u> Let n_0 be an integer.
Consider the infinite series

$$\sum_{n=n_0}^{\infty} a_n = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \cdots$$

where each a_n is a complex number.
From this series we create a sequence $(s_k)_{k=1}^{\infty}$ of <u>partial sums</u> defined by

$$s_k = \sum_{n=n_0}^{n_0+k-1} a_n = \underbrace{a_{n_0} + a_{n_0+1} + \cdots + a_{n_0+k-1}}_{\text{this is the sum of the first } k \text{ terms of the reies}}$$

Thus,

$$s_1 = a_{n_0}$$

$$s_2 = a_{n_0} + a_{n_0+1}$$

$$s_3 = a_{n_0} + a_{n_0+1} + a_{n_0+2}$$

$$\vdots \qquad \vdots$$

We say that $\sum_{n=n_0}^{\infty} a_n$ converges to the limit S, and we write $\sum_{n=n_0}^{\infty} a_n = S$, if $\lim_{k\to\infty} S_k = S$.

Otherwise, if $(S_k)_{k=1}^{\infty}$ diverges then we say that $\sum_{n=n_0}^{\infty} a_n$ diverges.

Ex: Consider the series

$$\sum_{n=0}^{\infty}\left(\frac{i}{3}\right)^n = 1 + \frac{i}{3} - \frac{1}{9} - \frac{i}{27} + \frac{1}{81} + \cdots$$

Does this converge?
What are the partial sums?

$S_1 = \boxed{1}$

$S_2 = \boxed{1 + i\frac{1}{3}}$

$S_3 = 1 + \frac{i}{3} - \frac{1}{9} = \boxed{\frac{8}{9} + i\frac{1}{3}}$

$S_4 = 1 + \frac{i}{3} - \frac{1}{9} - \frac{i}{27} = \boxed{\frac{8}{9} + i\frac{8}{27}}$

$\vdots \qquad \vdots$

Geometric sum formula

Algebra formula: If $z \neq 1$, then

$$1 + z + z^2 + \dots + z^k = \frac{1 - z^{k+1}}{1 - z}$$

proof:

$$(1 - z)(1 + z + z^2 + \dots + z^k)$$

$$= 1 + \cancel{z} + \cancel{z^2} + \dots + \cancel{z^k} - \cancel{z} - \cancel{z^2} - \dots - \cancel{z^k} - z^{k+1}$$

$= 1 - z^{k+1}$. Now divide by $1 - z$. $\blacksquare$

Back to our example.

$$S_k = 1 + \left(\frac{i}{3}\right) + \left(\frac{i}{3}\right)^2 + \dots + \left(\frac{i}{3}\right)^{k-1}$$

$$= \frac{1 - \left(\frac{i}{3}\right)^k}{1 - i/3}$$

$\boxed{z = \frac{i}{3}}$

Later today we will show:

If $\lim_{n\to\infty} |b_n| = 0$, then $\lim_{n\to\infty} b_n = 0$

Thus,

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1 - \left(\frac{i}{3}\right)^k}{1 - i/3}$$

$$= \frac{1 - \lim_{k\to\infty} \left(\frac{i}{3}\right)^k}{1 - i/3}$$

$$= \frac{1-0}{1-i/3} = \frac{1}{1-i/3}$$

$$\lim_{k\to\infty} \left|\left(\frac{i}{3}\right)^k\right| = \lim_{k\to\infty} \left|\frac{i}{3}\right|^k = \lim_{k\to\infty} \left(\frac{1}{3}\right)^k = 0$$

Calculus: $\lim_{k\to\infty} r^k = 0$ if $-1 < r < 1$.

$$= \frac{1}{1-\frac{i}{3}} \cdot \frac{1+\frac{i}{3}}{1+\frac{i}{3}}$$

$i^2 = -1$

$$= \frac{1+\frac{i}{3}}{1+\frac{1}{9}} = \frac{1+\frac{i}{3}}{10/9} = \boxed{\frac{9}{10} + i\frac{3}{10}}$$

Thus, $\sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n = \frac{9}{10} + i\frac{3}{10}$

It converges!

Note: For proofs and future definitions we will write our sequences and series starting at 1 instead of n_0. But the results will still hold true if they started at any n_0.

Divergence Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series of complex numbers. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Contrapositive: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges proof: (HW 1)

Suppose $\sum_{n=1}^{\infty} a_n$ converges to S.

Then,

$$\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty}\left[(a_1+a_2+\dots+a_n)-(a_1+a_2+\dots+a_{n-1})\right]$$

$$= \lim_{n\to\infty}\left[S_n - S_{n-1}\right]$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S$$

$$= 0$$

$\boxed{\text{DIVERGENCE!}}$

Theorem: Let $(b_n)_{n=1}^{\infty}$ be a sequence of complex numbers.

Then, $\lim_{n\to\infty} b_n = 0$ iff $\lim_{n\to\infty} |b_n| = 0$.

Proof:

($\Rightarrow$) Suppose $\lim_{n\to\infty} b_n = 0$.

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} b_n = 0$ there exists $N > 0$ where if $n \geq N$ then $|b_n - 0| < \varepsilon$.

Thus, if $n \geq N$, then

$$\big|\,|b_n| - 0\,\big| = \big|\,|b_n|\,\big| = |b_n| = |b_n - 0| < \varepsilon$$

So, $\lim_{n\to\infty} |b_n| = 0$

($\Leftarrow$) Suppose $\lim_{n\to\infty} |b_n| = 0$.

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} |b_n| = 0$ there exists $N > 0$ where if $n \geq N$ then $||b_n| - 0| < \varepsilon$.

So if $n \geq N$ then

$$|b_n - 0| = |b_n| = ||b_n|| = ||b_n| - 0| < \varepsilon$$

Thus, $\lim_{n\to\infty} b_n = 0$

$\blacksquare$

Ex: (Geometric Series)

Let $z \in \mathbb{C}$.

Consider the series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

For what z does this series converge and if it converges, what do it converge to?

The partial sums are

$$s_1 = 1$$

$$s_2 = 1 + z$$

$$s_3 = 1 + z + z^2$$

$$\vdots \qquad \vdots$$

And,

$$S_k = 1 + z + z^2 + \cdots + z^{k-1}$$

Ergo,

$$S_k = \begin{cases} k & , \text{ if } z = 1 \\ \dfrac{1 - z^k}{1 - z} & , \text{ if } z \neq 1 \end{cases}$$

<u>case 1: Suppose $|z| < 1$</u>

Since $|z| < 1$ we know

$$\lim_{k \to \infty} |z^k| = \lim_{k \to \infty} |z|^k = 0$$

So by our previous theorem

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-z^k}{1-z}$$

$$= \frac{1 - \lim_{k\to\infty} z^k}{1-z}$$

$$= \frac{1-0}{1-z} = \frac{1}{1-z}$$

<u>case 2:</u> Suppose $|z|>1$.

Then, $\lim_{k\to\infty} |z^k| = \lim_{k\to\infty} |z|^k = \infty$

So, $\lim_{k\to\infty} z^k \neq 0$.

Hence, by the divergence test $\sum_{n=1}^{\infty} z^n$ diverges in this case.

<u>case 3:</u> Suppose $|z|=1$.

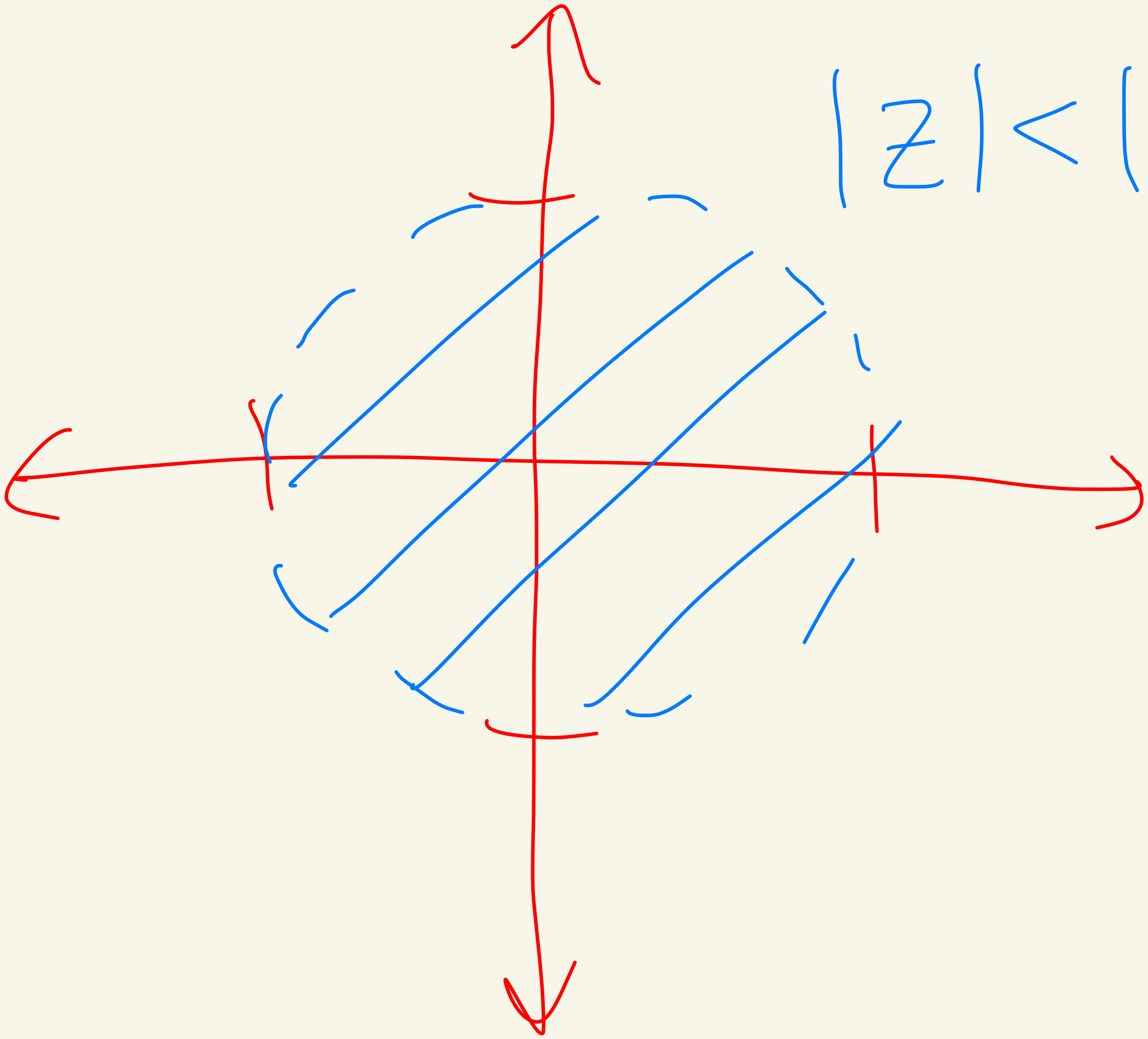
Then, $\lim\limits_{k\to\infty} |z^k| = \lim\limits_{k\to\infty} |z|^k = 1$

So, $\lim\limits_{k\to\infty} z^k \neq 0$.

Hence by the divergence test

$\sum\limits_{n=1}^{\infty} z^n$ diverges in this case.

$$\boxed{\begin{array}{l} \underline{\text{Conclusion:}} \ \sum\limits_{n=0}^{\infty} z^n \text{ converges} \\ \text{iff } |z|<1. \\ \text{If } |z|<1, \text{ then } \sum\limits_{n=0}^{\infty} z^n = \dfrac{1}{1-z} \end{array}}$$



$|z| < 1$