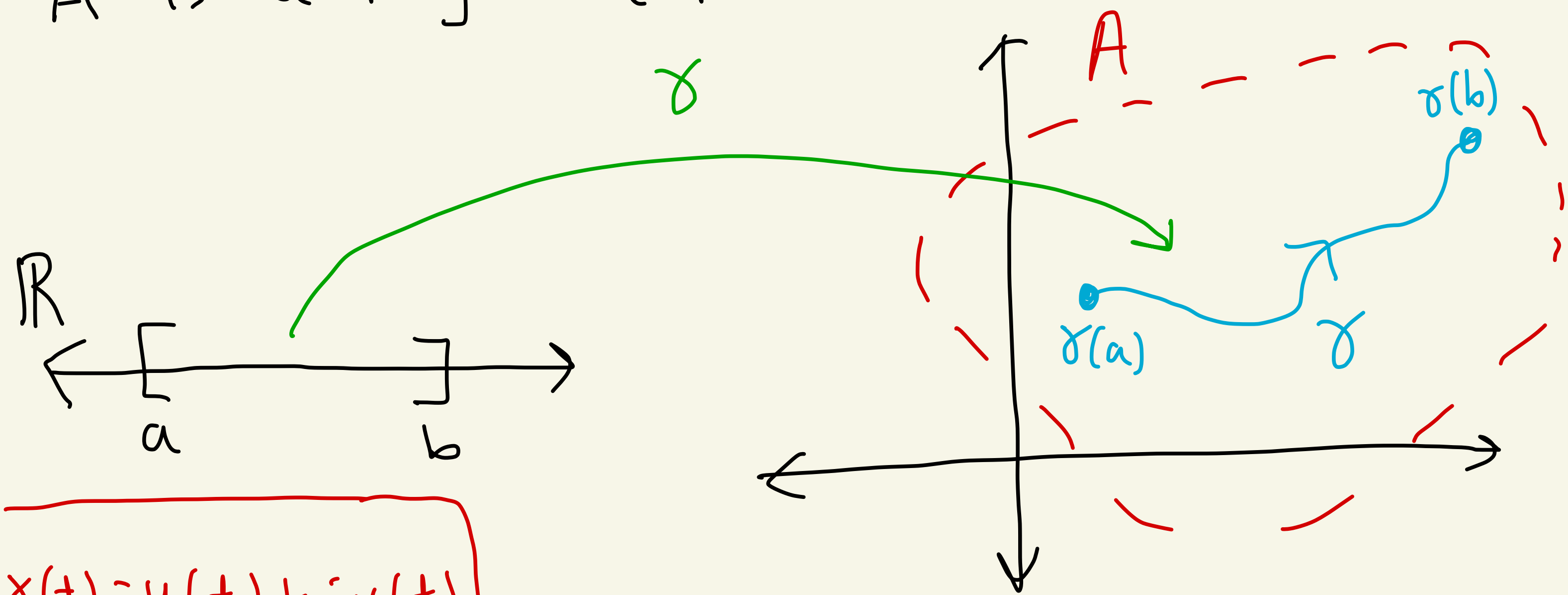
Math 5680

2/13/23

Theorem: Let $\gamma : [a,b] \to A$ be a piecewise-smooth curve where A is a region (open and path-connected)



$$\boxed{\begin{array}{c}\gamma(t) = u(t) + iv(t) \\ a \le t \le b\end{array}}$$

Let $f_n : A \to \mathbb{C}$ be continuous functions on A for $n \geq 1$.

Suppose $f_n \to f$ uniformly on A.

Then, $\displaystyle \lim_{n\to\infty} \int_\gamma f_n = \int_\gamma \lim f_n = \int_\gamma f$

proof:

Since $f_n \to f$ uniformly on A and each f_n is continuous on A, by a previous theorem f is continuous on A.

Since f_n and f are continuous on A we know $\int_\gamma f_n$ and $\int_\gamma f$ exist.

Let $\varepsilon > 0$.

Since $f_n \to f$ uniformly on A there exists $N > 0$ where if $n \geqslant N$ then $|f_n(z) - f(z)| < \frac{\varepsilon}{\text{length}(\gamma)}$ for all $z \in A$.

[length(γ) is arclength(γ)]

Thus, if $n \geq N$, then

$$\Big| \underbrace{\int_\gamma f_n(z)\,dz}_{\text{sequence}} - \underbrace{\int_\gamma f(z)\,dz}_{\text{limit}} \Big|$$

$$= \Big| \int_\gamma \big(f_n(z) - f(z)\big)\,dz \Big|$$

$$< \frac{\varepsilon}{\text{length}(\gamma)} \cdot \text{length}(\gamma)$$

$$= \varepsilon.$$

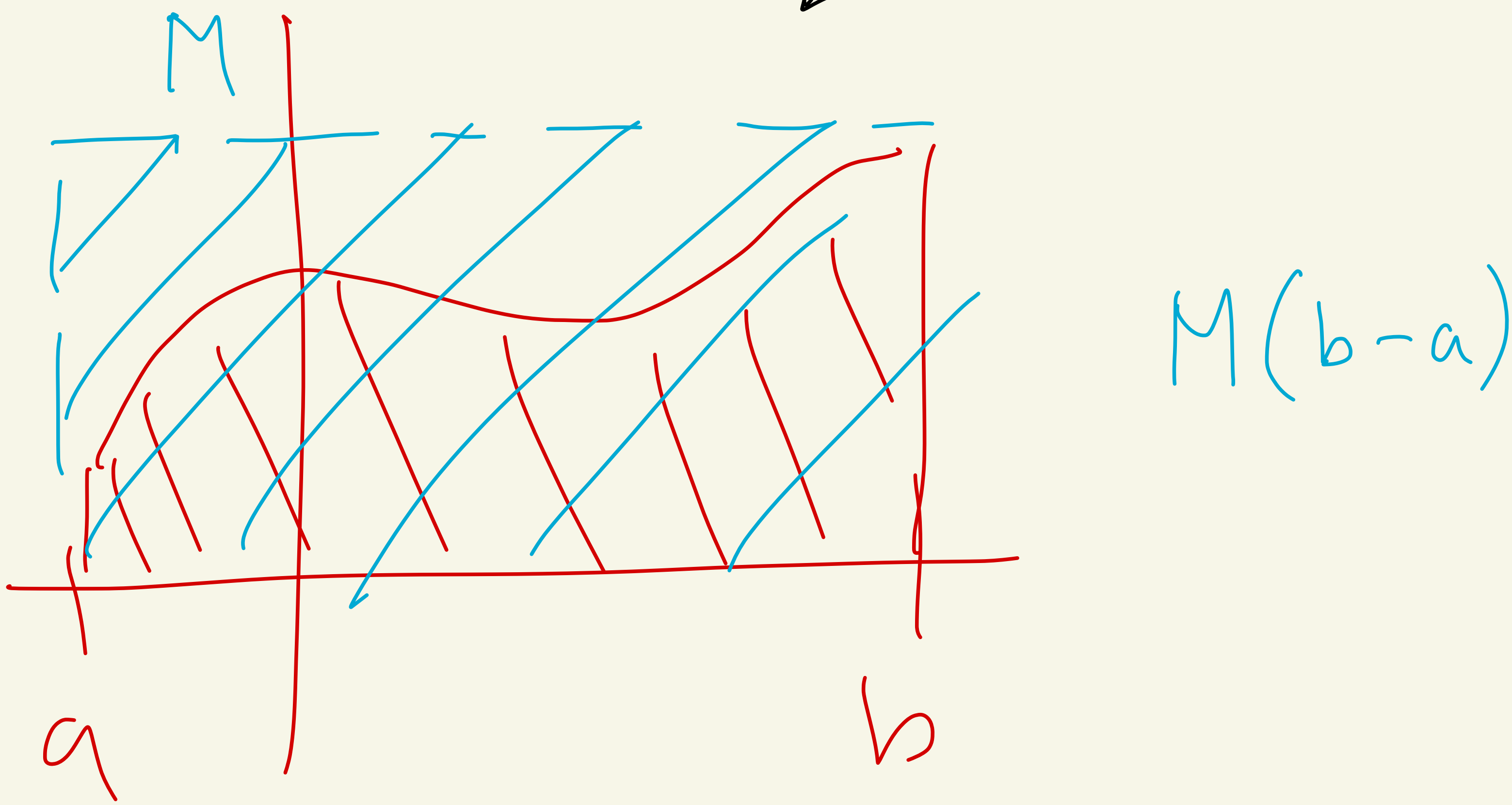
So $\lim\limits_{n\to\infty} \int_\gamma f_n = \int_\gamma f$. ∎

4680 Thm

If $|g(z)| \leq M$ for all z on γ, then

$$\Big|\int_\gamma g(z)\,dz\Big| \leq M \cdot \text{length}(\gamma)$$

4650 VERSION



<u>Corollary:</u> Let $A \subseteq \mathbb{C}$ be a region (open and path-connected). Let $\gamma : [a,b] \to A$ be a piecewise-smooth curve in A. Suppose $g_k : A \to \mathbb{C}$ is continuous on A for each $k \geq 1$.

Suppose $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A.

means: $f_n(z) = \sum_{k=1}^{n} g_k(z)$ ← partial sums

$$f(z) = \sum_{k=1}^{\infty} g_k(z)$$

$f_n \to f$ uniformly on A

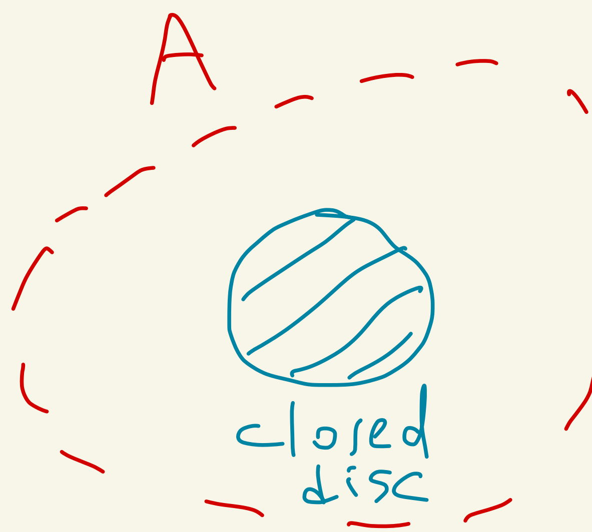
Then, $$\int_\gamma \left(\sum_{k=1}^{\infty} g_k(z)\right) dz = \sum_{k=1}^{\infty} \left(\int_\gamma g_k(z)\,dz\right)$$

Proof: See notes online.

Apply previous thm to f_n, f given above in red. $\blacksquare$

Theorem (Analytic Convergence Thm)

① Let A be an open set in $\mathbb{C}$. Let (f_n) be a sequence of analytic functions defined on A.
If $f_n \to f$ uniformly on every closed disc contained in A, then f is analytic.
Furthermore, $f_n' \to f'$ pointwise in A and uniformly on every closed disc in A.



② If (g_k) is a sequence of analytic functions defined on an open set A, and $g(z) = \sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every closed disc in A

then $g(z)$ is analytic on A and

$$g'(z) = \sum_{k=1}^{\infty} g_k'(z)$$ pointwise on A

and uniformly on every closed disc in A.

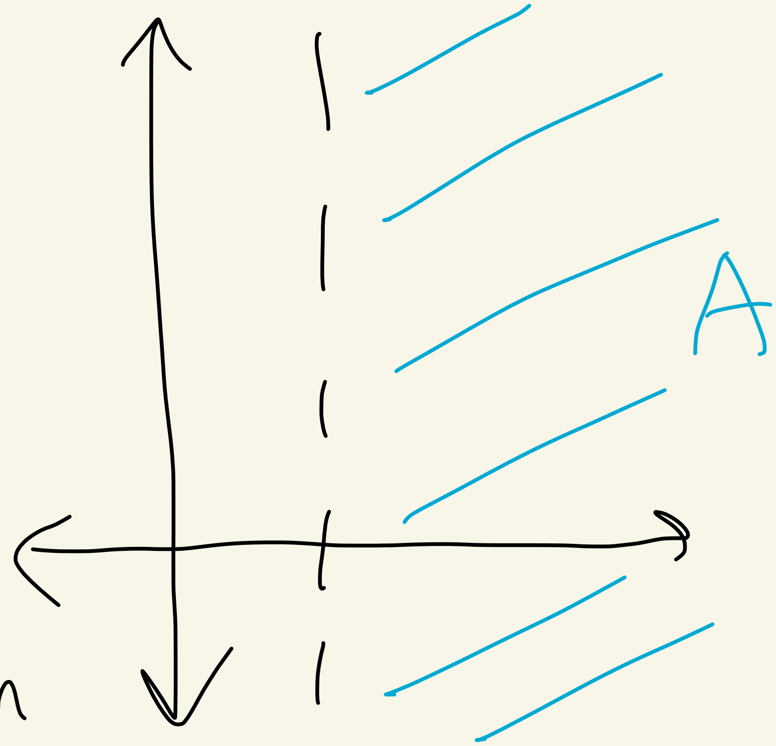
Ex: Let $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$

be the Riemann zeta function.

We know $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges

pointwise on

$A = \{ z \mid \text{Re}(z) > 1 \}$



Let's use the analytic convergence theorem on $\zeta(z)$.

Note $\frac{1}{n^z}$ is analytic on A

Recall: $(n^{-z})' = \log(n) \cdot n^{-z} \cdot (-1)$

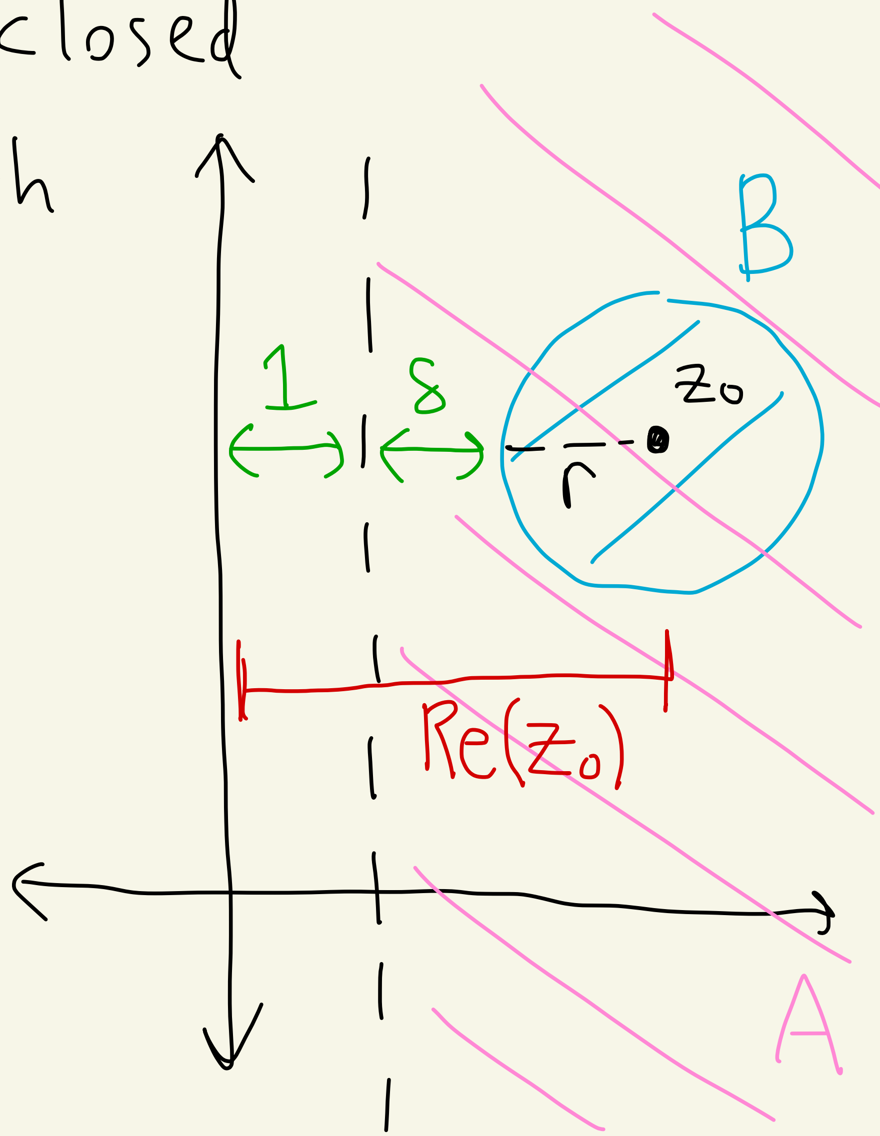
$= -\log(n) \cdot n^{-z}$

$\boxed{(a^z)' = \log(a) \cdot a^z} \; \boxed{a \neq 0}$

Let B be a closed disc in A, with center z_0 and radius r.

Let

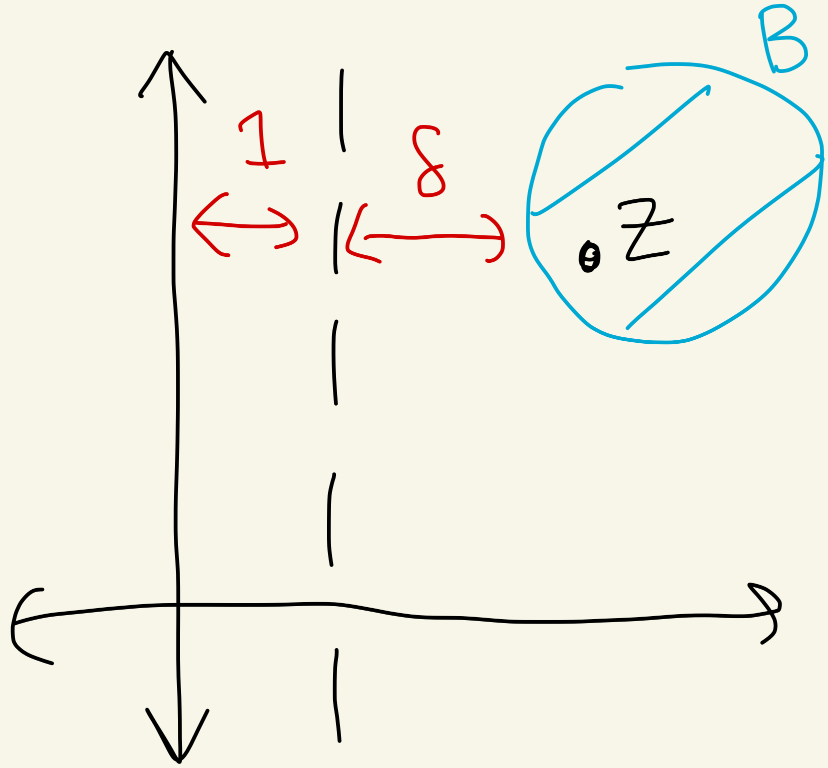
$\delta = \mathrm{Re}(z_0) - 1 - r$



Let $z \in B$.

Then,

$$\mathrm{Re}(z) \geq 1 + \delta$$



So if $z = x + iy$

then $x \geq 1 + \delta$.

So,

$$\left|\frac{1}{n^z}\right| = \left|n^{-z}\right| = \left|e^{-z\log(n)}\right|$$

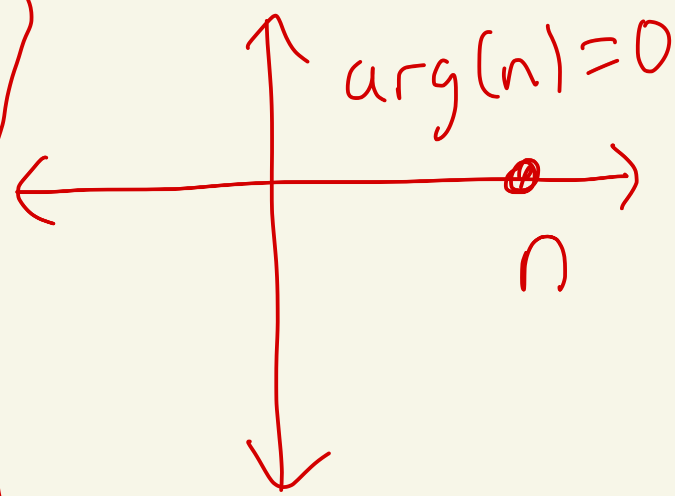
$$= \left|e^{-(x+iy)[\ln(n) + i\arg(n)]}\right|$$

$$= \left|e^{(-x-iy)\ln(n)}\right|$$

$$= \left|e^{-x\ln(n)} e^{i(-y\ln(n))}\right|$$

$$= \left|e^{-x\ln(n)}\right| \underbrace{\left|e^{i(-y\ln(n))}\right|}_{1}$$

$-\pi \leq \arg(n) < \pi$



$$= \left|e^{-x\ln(n)}\right|$$

$$\boxed{\begin{gathered}|e^{i\theta}| = 1 \\ \theta \in \mathbb{R}\end{gathered}}$$

$$= e^{\ln(n^{-x})}$$

$$= n^{-x} = \frac{1}{n^x} \leq \frac{1}{n^{1+\delta}}$$

$$\boxed{\begin{gathered}x \geq 1+\delta \\ n^x \geq n^{1+\delta}\end{gathered}}$$

Set $M_n = \dfrac{1}{n^{1+\delta}}$

(i) Note that if $z \in B$

then $\left|\frac{1}{n^z}\right| \leq M_n$ for $n \geq 1$

(ii) And $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$

converges since $1+\delta > 1$

$\boxed{\delta > 0}$

By the Weierstrass M-test

$\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely and

uniformly on B.

Thus, by the analytic convergence

theorem $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is analytic on A

and

$$\rho'(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n^z} \right)' = \sum_{n=1}^{\infty} \frac{-\log(n)}{n^z}$$

$$= \sum_{n=2}^{\infty} \frac{-\log(n)}{n^z}$$