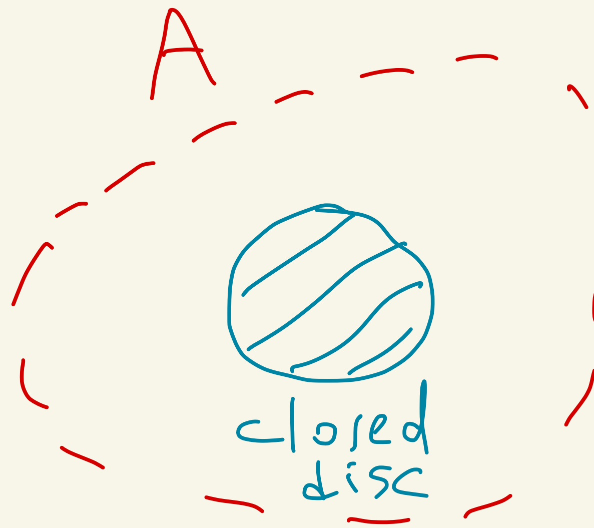
Math 5680
2/15/23

Theorem (Analytic Convergence Thm)

(1) Let A be an open set in $\mathbb{C}$.
Let (f_n) be a sequence of analytic functions defined on A.
If $f_n \to f$ uniformly on every closed disc contained in A,
then f is analytic.
Furthermore, $f_n' \to f'$ pointwise in A and uniformly on every closed disc in A.



(2) If (g_k) is a sequence of analytic functions defined on an open set A, and $g(z) = \sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every closed disc in A

then $g(z)$ is analytic on A and

$$g'(z) = \sum_{k=1}^{\infty} g_k'(z)$$ pointwise on A

and uniformly on every closed disc in A.

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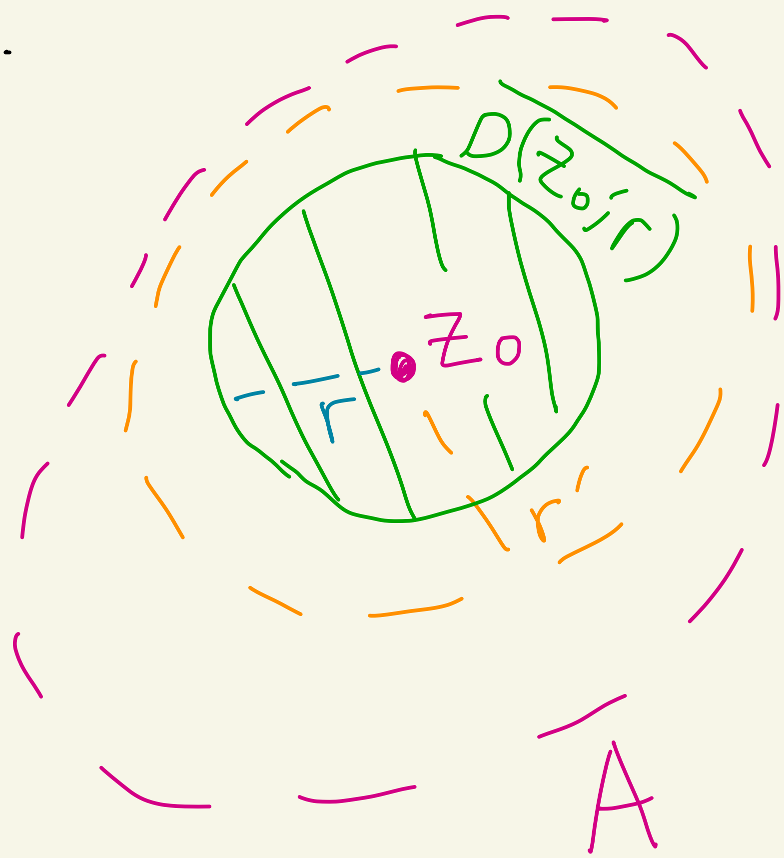
proof:

① Let $z_0 \in A$.

Our goal is to show that f is analytic at z_0.



Since A is open we can find $r' > 0$ where $D(z_0; r')$ is contained in A.

Pick some $r < r'$.

Then

$$\overline{D(z_0;r)} = \{z \mid |z-z_0| \leq r\}$$

is contained in A.

Since $f_n \to f$ uniformly on $\overline{D(z_0;r)}$ by assumption, this implies $f_n \to f$ uniformly on

$$D(z_0;r) = \{z \mid |z-z_0| < r\}$$

Since each f_n is continuous on $D(z_0;r)$, by a previous thm since $f_n \to f$ uniformly on $D(z_0;r)$ we know that

f_n analytic on A, and thus continuous on A

f is continuous on $D(z_0;r)$.

Let T be any triangular path inside of $D(z_0;r)$.

Since each f_n is analytic on T and inside of T, by Cauchy's theorem (Math 4680) we know

$$\int_T f_n = 0 \quad \text{for all } n.$$

By a previous theorem we have

$$0 = \lim_{n\to\infty} \underbrace{\int_T f_n}_{0} = \int_T \lim_{n\to\infty} f_n = \int_T f$$

Thus, $\int_T f = 0$ for any triangular path inside of $D(z_0; r)$.

By Morera's theorem, f is analytic in $D(z_0; r)$.

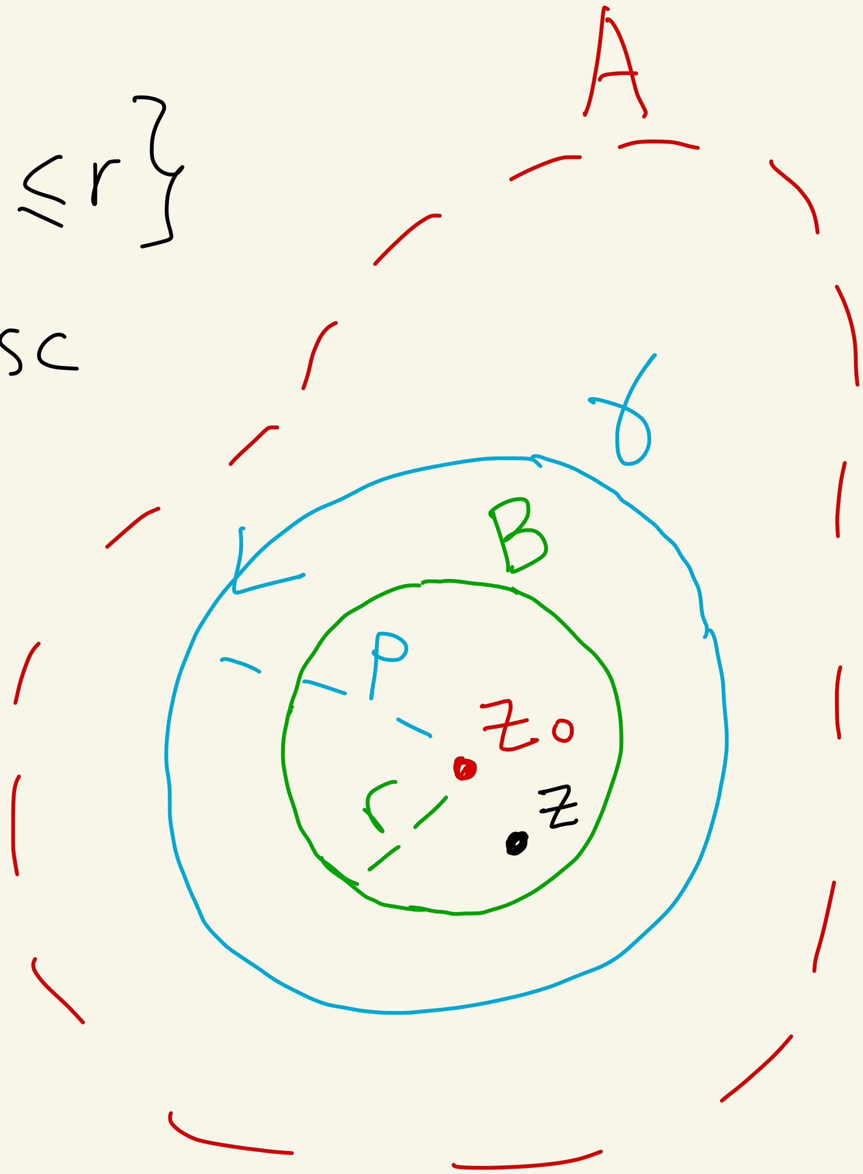
So, f is analytic at z_0.

We now show that $f_n' \to f'$ uniformly on closed discs in A.

Let

$B = \{ z \mid |z - z_0| \le r \}$

be a closed disc in A, where $r > 0$ and $z_0 \in A$.



By HW 2 problem A, we can choose $\rho > r$ such that γ is a circle contained in A of radius ρ that contains B in its interior.

Orient γ to be counterclockwise.

For any $z \in B$ we have

$$f_n'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w)}{(w-z)^2}\,dw$$

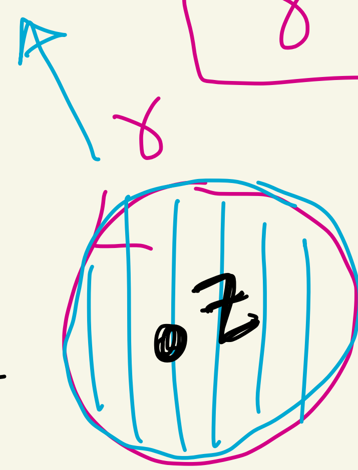
$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}\,dw$$

4680 Cauchy Integral Thm

Let $\varepsilon > 0$.

Since $f_n \to f$ uniformly on

$\overline{D(z_0;\rho)} = \{z \mid |z - z_0| \le \rho\}$ ← inside and on γ



there exists $N > 0$ where if $n \geq N$ then

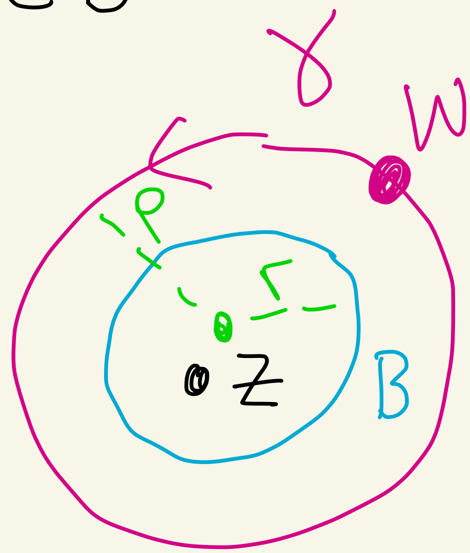
$$|f_n(z) - f(z)| < \varepsilon \cdot \frac{(\rho - r)^2}{\rho}$$

for all $z \in \overline{D(z_0;\rho)}$

If w is on γ and $z \in B$ then

$$|w-z| \geq \rho - r$$

ie, $$\frac{1}{|w-z|} \leq \frac{1}{(\rho - r)}$$



Thus, if $n \geq N$ and $z \in B$ then

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2} dw \right|$$

$$= \frac{1}{2\pi} \left| \int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2} dw \right|$$

$2\pi\rho$

$$< \frac{1}{2\pi} \cdot \frac{\varepsilon \frac{(\rho - r)^2}{\rho}}{(\rho - r)^2} \cdot \text{length}(\gamma)$$

w is on γ, z is in B, $n \geq N$

$$\left| \frac{f_n(w) - f(w)}{(w-z)^2} \right| = \frac{|f_n(w) - f(w)|}{|w-z|^2}$$

$$< \frac{\varepsilon \cdot \frac{(\rho - r)^2}{\rho}}{(\rho - r)^2}$$

$$= \varepsilon.$$

So, $f_n' \to f'$ uniformly on B.

② Set $f_n = \sum_{k=1}^{n} g_k$ and $f = \sum_{k=1}^{\infty} g_k$

and apply ① $\blacksquare$