Math 5680
2/20/23

Def: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

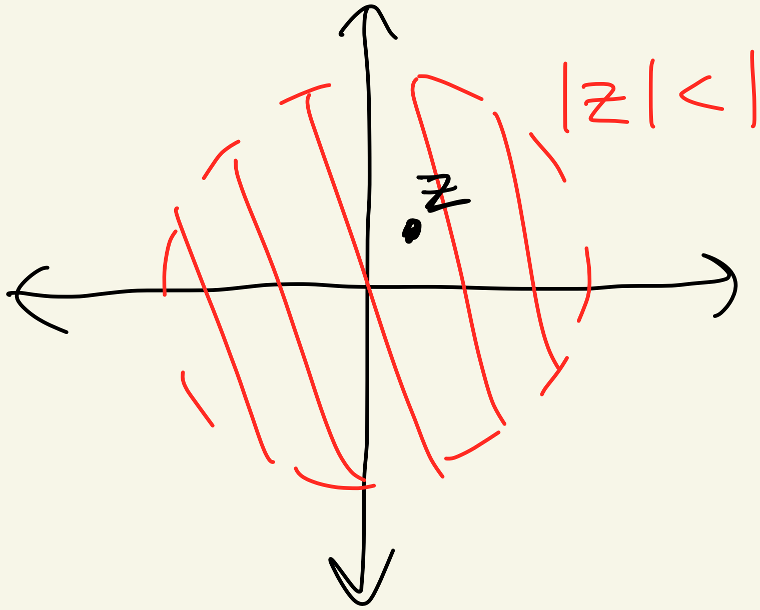
where z_0, a_n are constants in $\mathbb{C}$

We say that the power series is **centered** at z_0.

Ex: $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$

is centered at $z_0 = 0$

This series converges when $|z|<1$ to $\frac{1}{1-z}$



Ex: $\sum_{n=0}^{\infty}(-1)^n(z-1)^n$

$$= 1-(z-1)+(z-1)^2-(z-1)^3+\cdots$$

is centered at $z_0=1$

What will happen is this

series will have a disc centered at $z_0 = 1$ that it converges on.

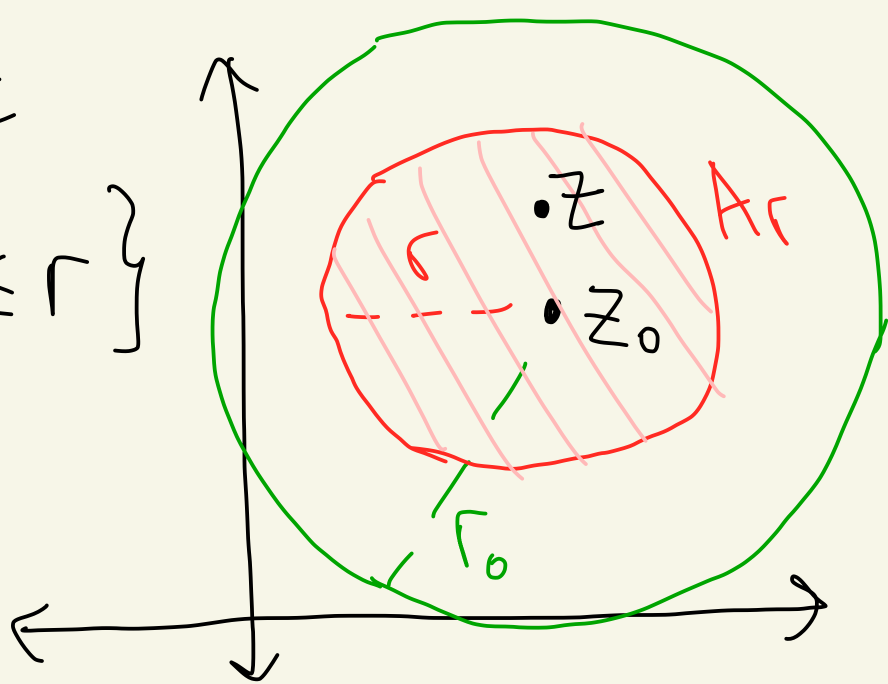
Lemma: (Abel-Weierstrass lemma)

Let $r_0 \in \mathbb{R}$, $a_n \in \mathbb{C}$, $n \geq 0$.

Suppose that $r_0 > 0$ and there exists $M > 0$ where $M \in \mathbb{R}$ such that $|a_n| r_0^n \leq M$, $\forall n \geq 0$.

Then for $r < r_0$, the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly and absolutely on the closed disc

$$A_r = \{ z \mid |z - z_0| \leq r \}$$



<u>proof:</u> Suppose $r_0 > 0$ and $|a_n| r_0^n \le M$ for all $n \geq 0$.

Let $r < r_0$.

Let $z \in A_r = \{ z \mid |z - z_0| \le r \}$.

Then,

$$
\begin{aligned}
|a_n (z - z_0)^n| &= |a_n| |z - z_0|^n \\
&\le |a_n| r^n \\
&= |a_n| r_0^n \cdot \left(\frac{r}{r_0}\right)^n \\
&\le M \left(\frac{r}{r_0}\right)^n
\end{aligned}
$$

$z \in A_r$

Let $M_n = M \left(\frac{r}{r_0}\right)^n$.

Note that

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} M\left(\frac{r}{r_0}\right)^n = M \frac{1}{1-\frac{r}{r_0}}$$

$r < r_0$

$\frac{r}{r_0} < 1$

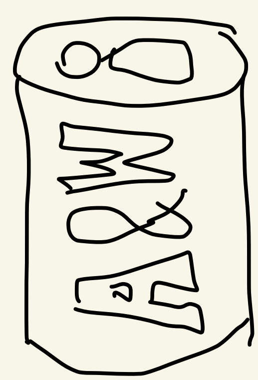
That is, $\sum_{n=0}^{\infty} M_n$ converges.

So, by the Weierstrass M-test

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely

and uniformly on A_r.

yummy
yummy
root beer

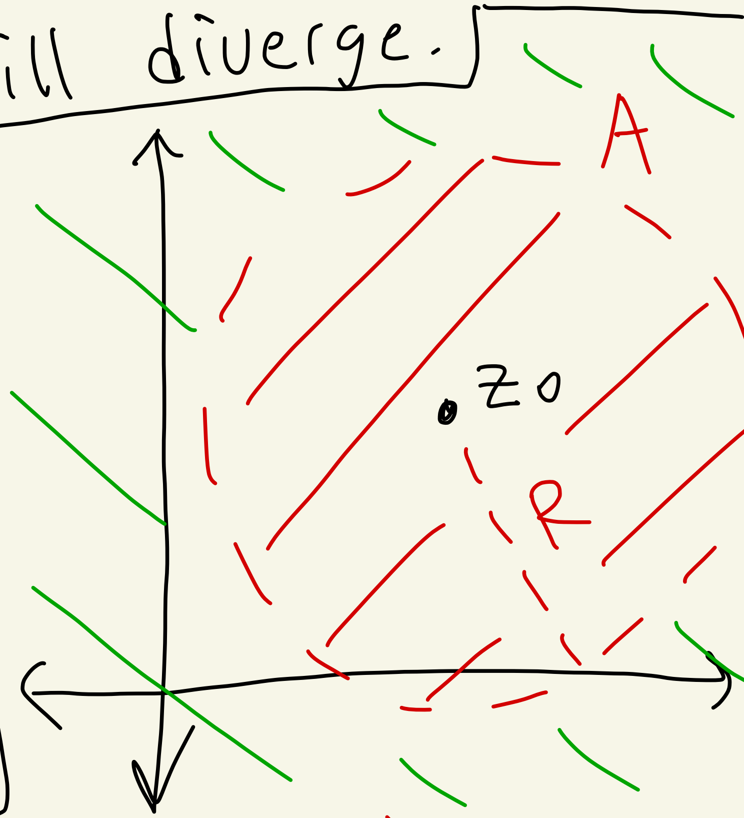


Theorem (Power Series Convergence)

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series.

Then there exists a unique number $R \geq 0$, possibly ∞, called the radius of convergence, such that if $|z-z_0| < R$ then the series will converge and if $|z-z_0| > R$ then the series will diverge.

Furthermore, the convergence is absolute and uniform on every closed disc contained in



$A = \{z \mid |z-z_0| < R\}$

converges on A
diverge outside of A
unknown on boundary

Proof: Let

$$S = \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} |a_n| r^n \text{ converges} \right\}$$

Let $R = \sup(S)$.

least upper bound of S

First suppose $R = 0$.

Then $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for all z where $|z - z_0| < \underbrace{R}_{0}$ since there are no such z.

Suppose $z_1 \in \mathbb{C}$ where $r_0 = |z_1 - z_0| > R$

We want to show that $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ diverges.

Suppose instead $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges,

Then $\lim_{n\to\infty} a_n(z_1 - z_0)^n = 0$ by the divergence theorem.

Thus,

$$\lim_{n\to\infty} |a_n| r_0^n = \lim_{n\to\infty} |a_n||z_1 - z_0|^n = 0$$

Since the sequence $\left(|a_n| r_0^n\right)_{n=0}^{\infty}$ converges, it is bounded.

So, $|a_n| r_0^n \leq M$ for some $M > 0$ where $n \geq 0$.

By the Abel-Weierstrass theorem,

if $0 < r < r_0$ then $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

converges absolutely for all

$z \in A_r = \{ z \mid |z-z_0| \leq r \}$.

So, $\sum_{n=0}^{\infty} |a_n| |z-z_0|^n$ converges

for all $z \in A_r$ with $0 < r < r_0$.

Thus, $\sum_{n=0}^{\infty} |a_n| r^n$ converges for

all r with $R = 0 < r < r_0$.

This contradicts that

$R = \sup(S)$ because we

would have $r \in S$ and $R < r$.

So, $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ diverges

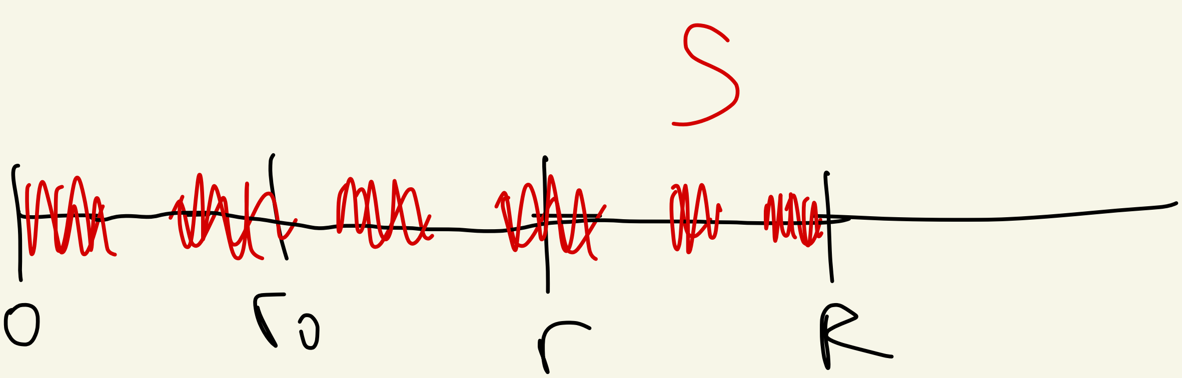
when $|z_1 - z_0| > R$.

<u>Now suppose $R = \sup(S) > 0$</u>

Let $0 < r_0 < R$.

If $\sum_{n=0}^{\infty} |a_n| r_0^n$ diverged then by the comparison test $\sum_{n=0}^{\infty} |a_n| r^n$ would diverge for $r_0 < r < R$.

Since $R = \sup(S)$ there exists $r \in S$ where $r_0 < r < R$.



Since $r \in S$ we know $\sum_{n=0}^{\infty} |a_n| r^n$ converges.

Thus, by above

$$\boxed{\text{if } 0 < r_0 < R, \text{ then } \sum_{n=0}^{\infty} |a_n| r_0^n \text{ converges.}}$$

Thus, $\lim_{n \to \infty} |a_n| r_0^n = 0$

Thus, $(|a_n| r_0^n)$ is a convergent sequence and hence is bounded.

So there exists $M > 0$ where $|a_n| r_0^n \leq M$ for all $n \geq 0$.

By the Abel-Weierstrass theorem

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ converges}$$

uniformly and absolutely on

$$A_r = \{ z \mid |z-z_0| \leq r \}$$

for any $r < r_0 < R$.

Note if z satisfies $|z-z_0| < R$ then we can always find an r_0 and r such that $z \in A_r$ and $r < r_0 < R$.

So, $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges for all z with $|z-z_0| < R$.

Continued next time