Math 5680
2/22/23

We continue the proof from last time

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series.

We set $R = \sup(S)$ where

$$S = \left\{ r \geq 0 \;\middle|\; \sum_{n=0}^{\infty} |a_n| r^n \text{ converges} \right\}$$

For $R=0$ we showed that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges if $|z-z_0|<R$ and diverges if $|z-z_0|>R$.

Then we assumed $R>0$.

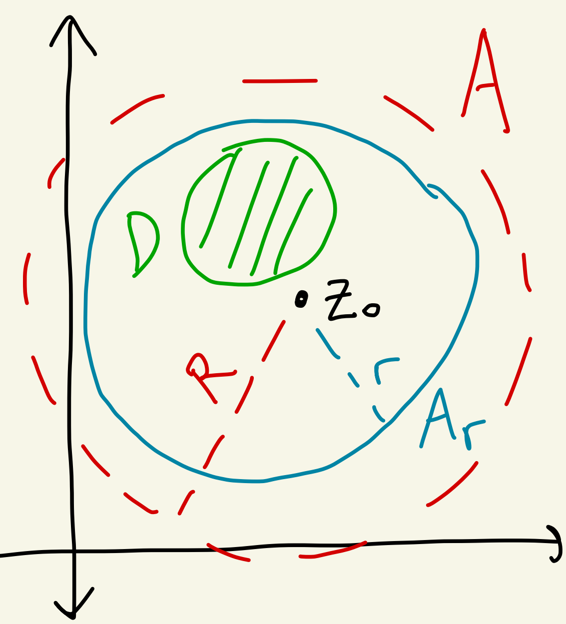
We showed $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

converges if $|z-z_0|<R$

continued....

Moreover, suppose D is some closed disc in $A = \{ z \mid |z-z_0|<R \}$.

There exists some r where $0<r<R$ where $D \subseteq A_r \subseteq A$ where $A_r = \{ z \mid |z-z_0| \leq r \}$

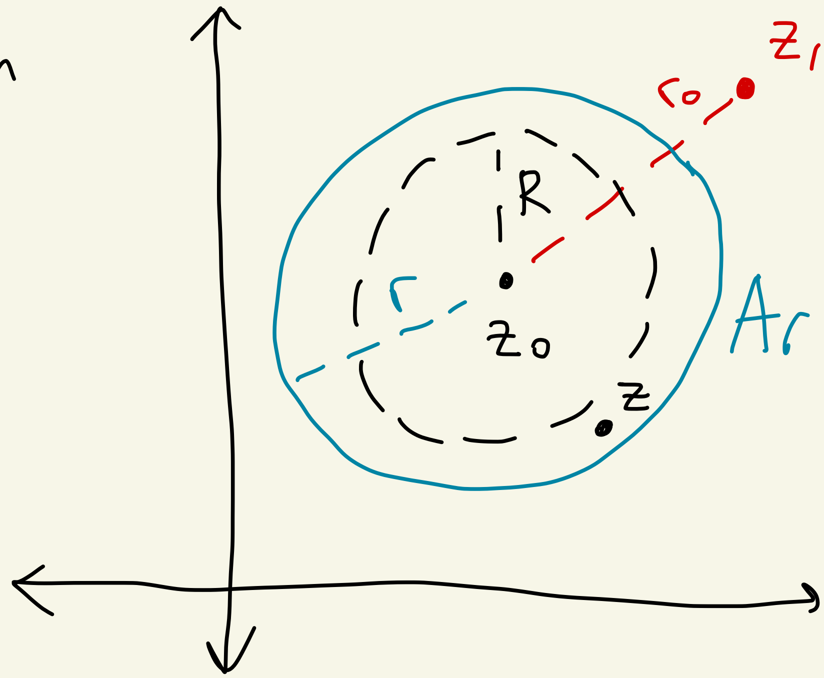


From last class we had that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

converges uniformly and absolutely on A_r

and thus also on D.

Now let's show the divergence part.



Suppose $z_1 \in \mathbb{C}$ with $r_0 = |z_1 - z_0| > R$ and $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges [we want it to diverge.]

Then, $\lim_{n\to\infty} a_n(z_1-z_0)^n = 0$

Since this sequence converges, its bounded.

So, $|a_n| r_0^n = |a_n(z_1-z_0)^n| \leq M$ where $M>0$ for all $n \geq 0$.

Thus, by the Abel-Weierstrass theorem if $R < r < r_0$ then

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely if $z \in A_r = \{z \mid |z-z_0| \leq r\}$

So, $\sum_{n=0}^{\infty} |a_n| |z-z_0|^n$ converges for all $z \in A_r$

Thus, $\sum_{n=0}^{\infty} |a_n| t^n$ converges for all t where $R < t < r$

This says $t \in S$ and $R < t$.

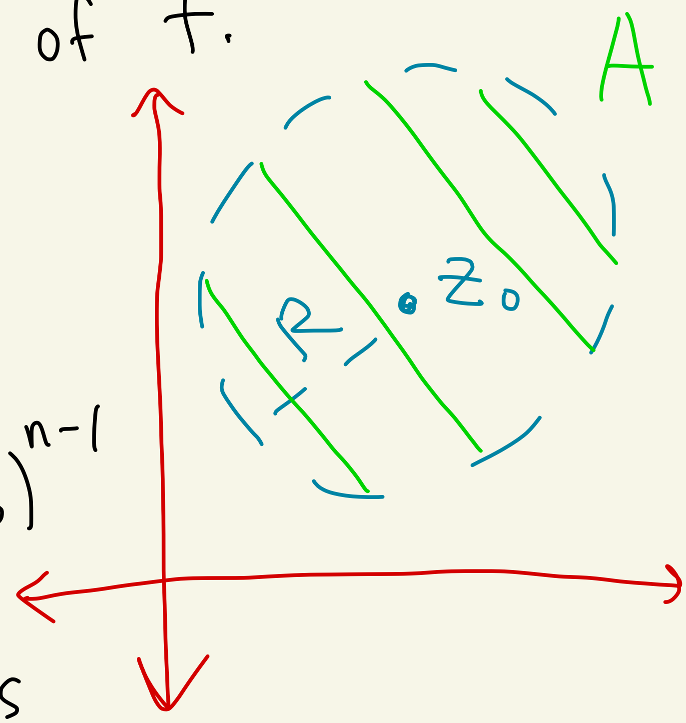
But $R = \sup(S)$.

Contradiction.

Thus, $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ diverges

when $|z_1 - z_0| > R$. $\blacksquare$

Theorem: Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series defined on $A = D(z_0; R)$ where R is the radius of convergence of f.



Then,

① f is analytic in A

② $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$

for $z \in A$. This series has the same radius of convergence R.

and

③ $a_n = \dfrac{f^{(n)}(z_0)}{n!}$

proof: The last theorem showed that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely and uniformly on any closed disc in A.

Thus, by the analytic convergence theorem f is analytic on A and $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ for all $z \in A$.

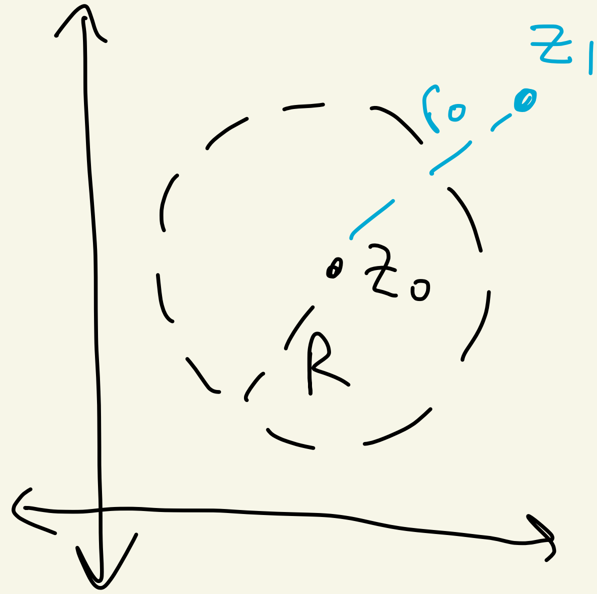
So the radius of convergence of $f'(z)$ is at least R.

Can it be bigger?

Suppose $z_1 \in \mathbb{C}$ with $r_0 = |z_1 - z_0| > R$ and $\sum_{n=1}^{\infty} n a_n (z_1 - z_0)^{n-1}$ converged.

[we want it to diverge]

Then,



$$\lim_{n\to\infty} n a_n (z_1 - z_0)^{n-1} = 0$$

Thus,

$$\lim_{n\to\infty} |n a_n r_0^{n-1}| = \lim_{n\to\infty} |n a_n (z_1 - z_0)^{n-1}| = 0$$

So, $(n|a_n| r_0^{n-1})_{n=1}^{\infty}$ converges.

So, its bounded.

That is, $n|a_n| r_0^{n-1} \leq M$ for all $n \geq 1$ for some $M > 0$.

Thus,

$$|a_n| r_0^n = |a_n r_0^n| = |n a_n r_0^{n-1}| \left|\frac{r_0}{n}\right| \leq M r_0$$

for all $n \geq 1$.

Let $M' = \max\{Mr_0, |a_0|r_0^n\}$.

Then, $|a_n|r_0^n \leq M'$ for $n \geq 0$.

By the A&W theorem

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ converges on}$$

$$A_r = \{z \mid |z-z_0| \leq r\}$$

for any r with

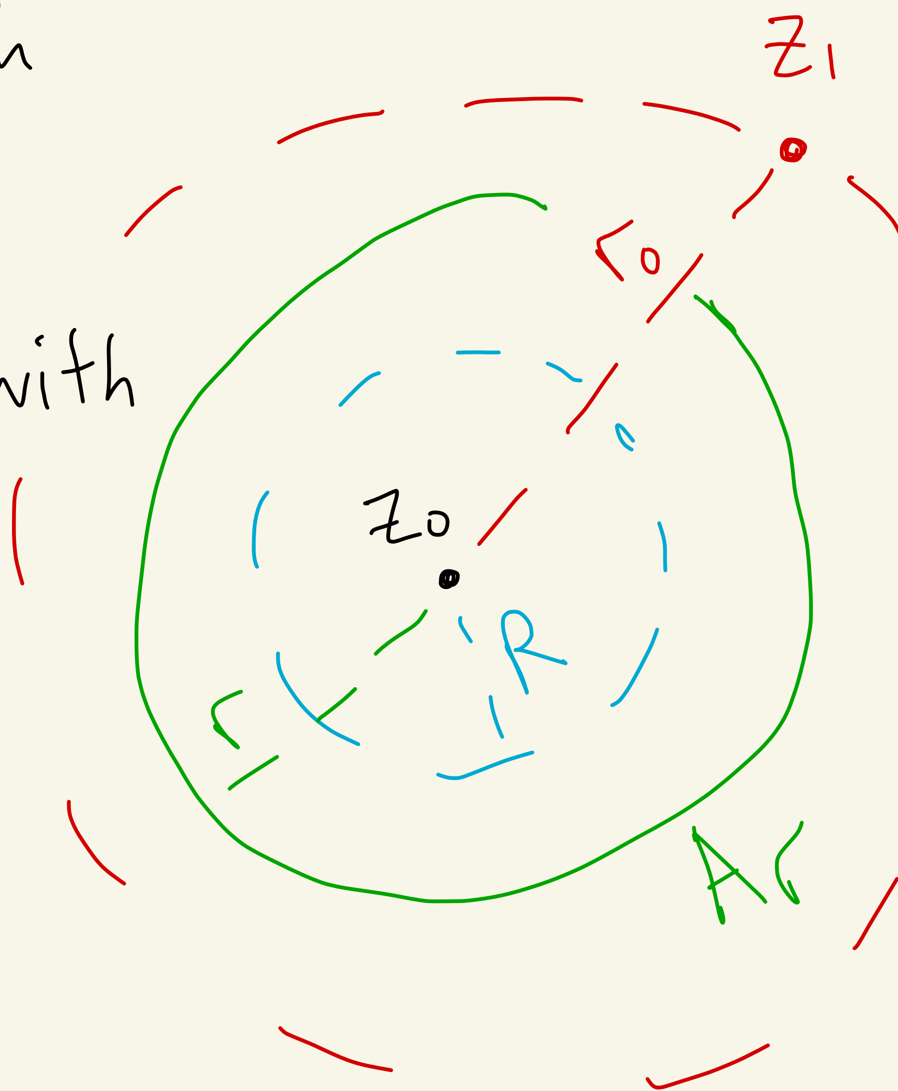
$0 < r < r_0$.

Pick some r with

$0 < R < r < r_0$.

This is a contradiction.

Thus,



$$\sum_{n=1}^{\infty} n a_n (z_1 - z_0)^{n-1} \text{ diverges.}$$

So, R is the radius of convergence of $f'(z)$.

You can keep applying the analytic convergence theorem and the above arguments to get power series for $f^{(n)}(z)$ they will all have radius of convergence R.

We get

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots$$

$$f''(z) = 2a_2 + 3\cdot 2\, a_3 (z - z_0) + 4\cdot 3\cdot a_4 (z - z_0)^2 + \cdots$$

In general,

$$f^{(k)}(z) = k!\, a_k + \sum_{n=k+1}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\, a_n (z - z_0)^{n-k}$$

Plug $z = z_0$ in to get

$$f^{(k)}(z_0) = k!\, a_k + \sum_{n=k+1}^{\infty} 0$$

So,

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

$\blacksquare$

Theorem (Uniqueness of Power Series)

If

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

for all $z \in D(z_0; r)$ with $r > 0$, then $a_n = b_n$ for all $n \geq 0$

proof:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = b_n \quad \blacksquare$$