Math 5680

2/27/23

**Ratio Test** Let $\sum_{k=1}^{\infty} b_k$.

be a series of complex numbers

Suppose that

$$r = \lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right|$$

exists.

① If $r < 1$, then $\sum_{k=1}^{\infty} b_k$ converges absolutely.

② If $r > 1$, then $\sum_{k=1}^{\infty} b_k$ diverges.

③ If $r = 1$, then the test is inconclusive, the series may converge or diverge.

__proof:__

__case 1:__ Suppose $0 \le r < 1$.

Let $r' \in \mathbb{R}$ with $r < r' < 1$.

Since we have a sequence of real numbers $\left|\frac{b_{k+1}}{b_k}\right|$ converging to $r$, there must exist $N > 0$ where if $k \geq N$, then $\left|\frac{b_{k+1}}{b_k}\right| < r'$.

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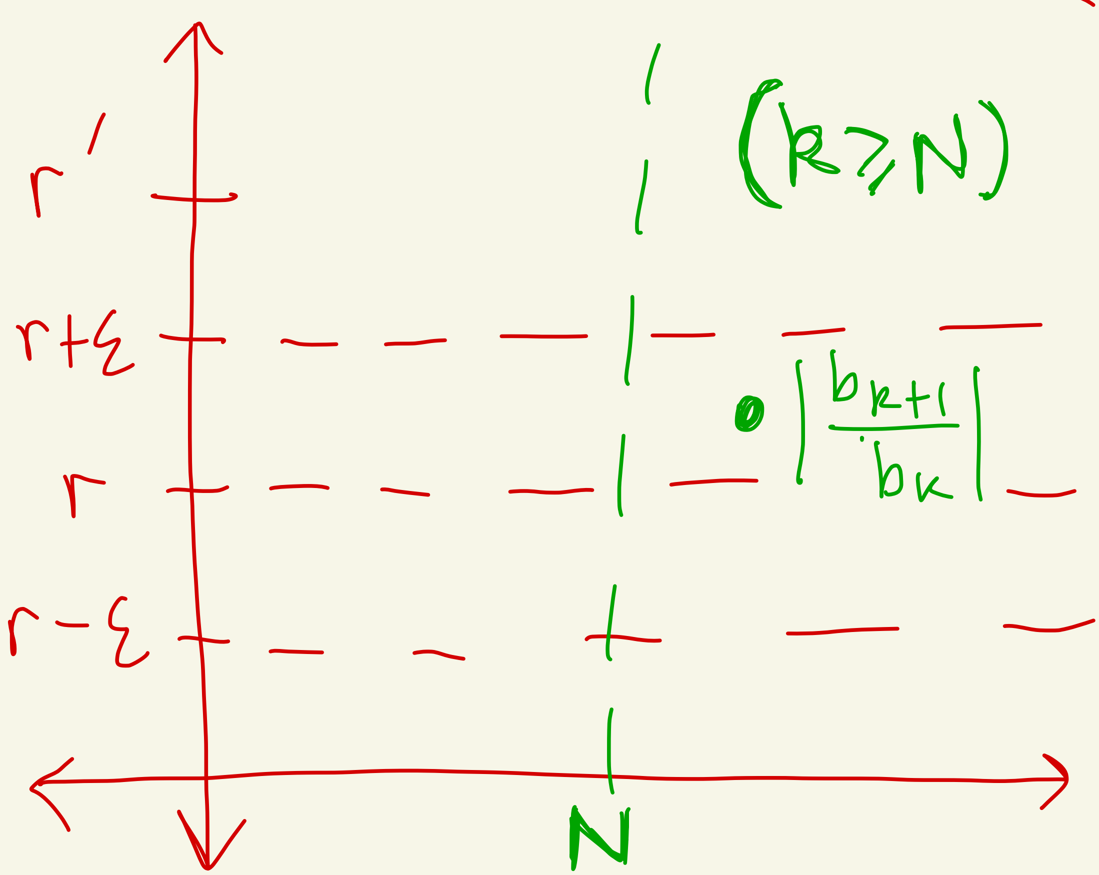
__Why?__

Set $\varepsilon = \frac{r'-r}{2}$

Then $\exists N > 0$ where if $k \geq N$ then

$\left|\frac{b_{k+1}}{b_k}\right| < r + \varepsilon$



$$= \frac{r'}{2} + \frac{r}{2} < \frac{r'}{2} + \frac{r'}{2} = r'$$

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Thus, if $k \geq N$ then

$$|b_k| < r'|b_{k-1}| < (r')^2 |b_{k-2}|$$

$$< \cdots < (r')^{k-N} |b_N|$$

The series

$$\sum_{k=N}^{\infty} (r')^{k-N} |b_N|$$

$$= |b_N| \sum_{k=N}^{\infty} (r')^{k-N}$$

$$= |b_N| \left(1 + r' + (r')^2 + (r')^3 + \cdots\right)$$

$$= |b_N| \frac{1}{1-r'}$$ (geometric sum)

Since $0 < r' < 1$,

Since $|b_k| < (r')^{k-N} |b_N|$

for all $k \geq N$ and $\sum_{k=N}^{\infty} (r')^{k-N} |b_N|$

converges, by the comparison test

(HW 1 #5), we know

$$\sum_{k=N}^{\infty} |b_k| \text{ converges.}$$

By HW 1 #2, this implies

$$\sum_{k=1}^{\infty} |b_k| \text{ converges.}$$

Thus, $\sum_{k=1}^{\infty} b_k$ converges absolutely.

<u>Case 2:</u> Suppose $r > 1$.

Choose $r' \in \mathbb{R}$ with $1 < r' < r$.

There must exist $N > 0$ where if $k \geq N$ then $\left|\frac{b_{k+1}}{b_k}\right| > r'$.

Why?

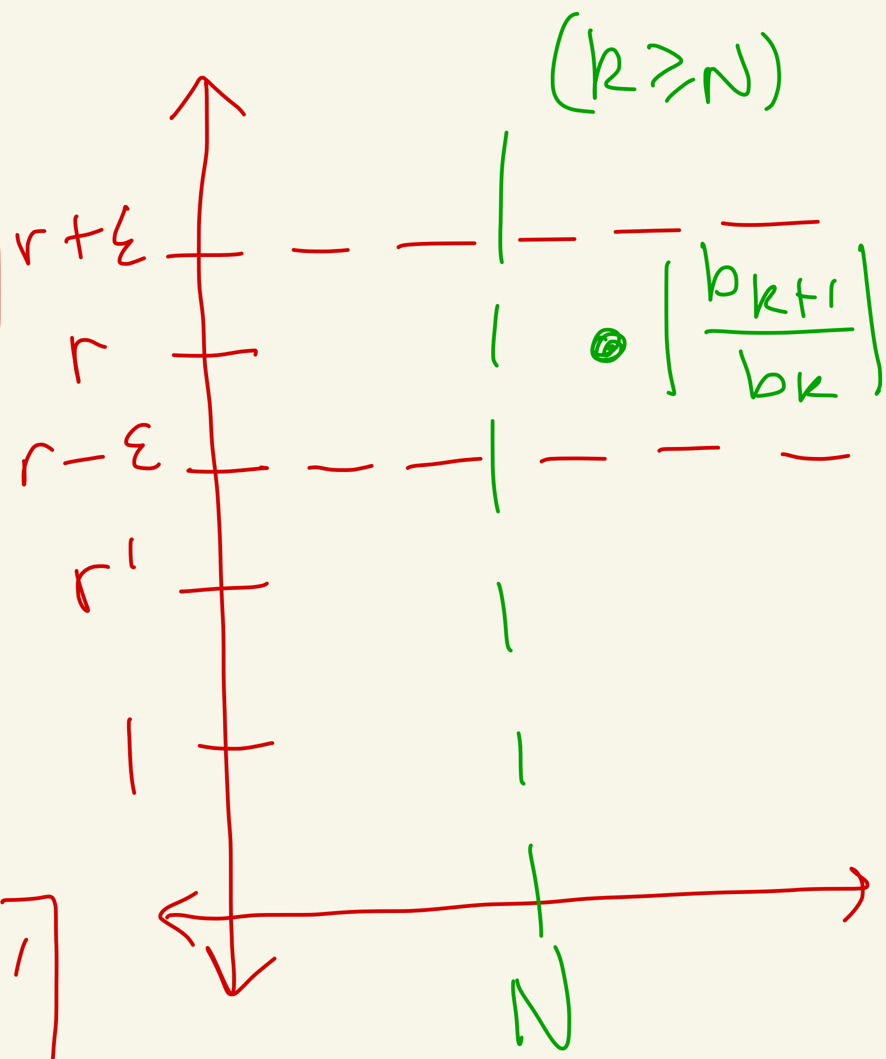
Let $\varepsilon = \frac{r - r'}{2}$

There exists $N > 0$ where if $k \geq N$

$$\left|\frac{b_{k+1}}{b_k}\right| > r - \varepsilon$$

$$= \frac{r}{2} + \frac{r'}{2}$$

$$> \frac{r'}{2} + \frac{r'}{2} = r'$$



Then,

$$|b_{N+p}| > r'|b_{N+p-1}| > (r')^2 |b_{N+p-2}|$$

$$> \cdots > (r')^p |b_N|$$

Thus,

$$\lim_{k\to\infty} |b_k| = \lim_{p\to\infty} |b_{N+p}|$$

$$> \lim_{p\to\infty} (r')^p \underbrace{|b_N|}_{\text{fixed \#}}$$

$$= \infty \quad (\text{since } 1 < r')$$

Thus, $\lim_{k \to \infty} b_k \neq 0$.

By the divergence test $\sum_{k=1}^{\infty} b_k$ diverges.

<u>Case 3:</u> Suppose $r = 1$.

The test is inconclusive.

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ has

$$\lim_{k \to \infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim \left| \frac{k}{k+1} \right| = 1 \leftarrow r$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>

For example, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has

$$\lim_{k\to\infty}\left|\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}}\right| = \lim_{k\to\infty}\left|\frac{k^2}{(k+1)^2}\right| = 1 \leftarrow r$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. $\blacksquare$

$\boxed{\text{HW 2 \#5}}$

Let $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!\,z^n}$

$A = \mathbb{C} - \{0\}$

(a) Show $g$ is analytic on $A$

(b) Find a formula for $g'$ on $A$.

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<u>proof</u>: We use the analytic convergence theorem.

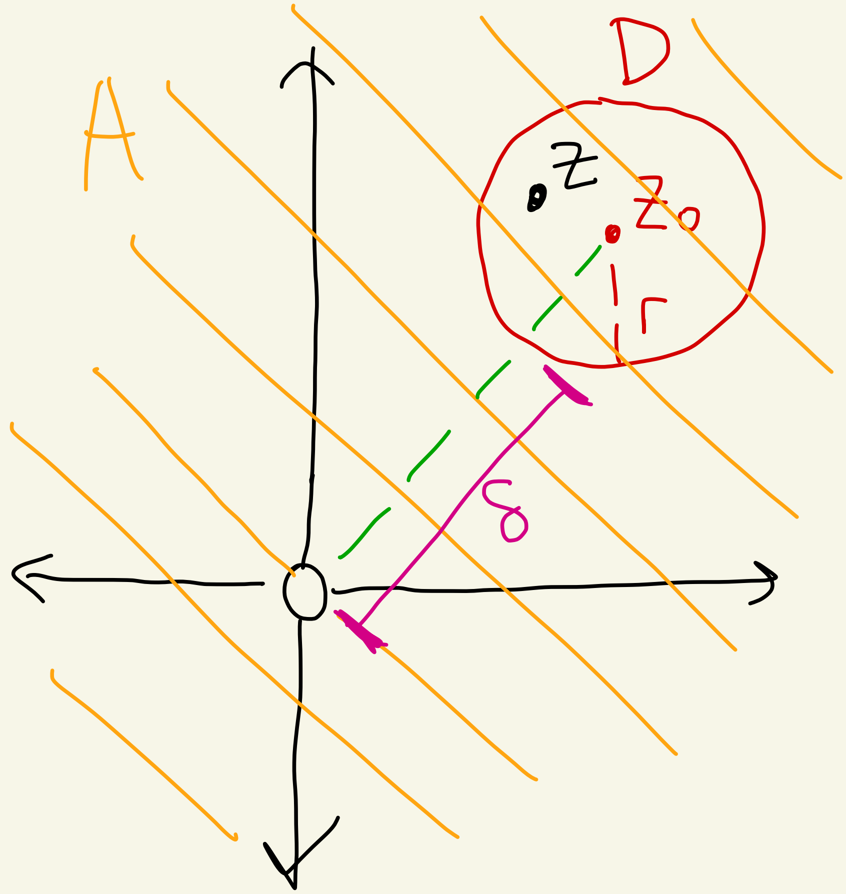
Let $D$ be a closed disc in $A$.

Let $D$ have center $z_0$ and radius $r$.

So, $D = \{z \mid |z - z_0| \le r\}$

Let

$\delta = |z_0| - r > 0$




Claim: If $z \in D$, then $|z| \geq \delta$.

pf of claim: Let $z \in D$.

Then, $|z - z_0| \leq r$.

Thus,

$$
\begin{aligned}
|z_0| &= |z_0 - z + z| \\
&\leq |z_0 - z| + |z| \\
&= |z - z_0| + |z| \\
&= r + |z|.
\end{aligned}
$$

So, $\underbrace{|z_0| - r}_{\delta} \leq |z|$.

Thus, $\delta \leq |z|$. $\boxed{\text{claim}}$

$\boxed{|z| \geq \delta}$

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Thus, if $z \in D$, then

$$\left| \frac{1}{n!} \cdot \frac{1}{z^n} \right| = \frac{1}{n!} \cdot \frac{1}{|z|^n} \leq \underbrace{\frac{1}{n!} \frac{1}{\delta^n}}_{M_n}$$

Let $M_n = \frac{1}{n!} \frac{1}{\delta^n}$.

Does $\sum_{n=1}^{\infty} M_n$ converge?

Let's use the ratio test

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}\frac{1}{\delta^{n+1}}}{\frac{1}{n!}\frac{1}{\delta^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\cdot\frac{\delta^n}{\delta^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)\delta}\right|$$

$$\boxed{(n+1)! = (n+1)\cdot[n!]}$$

$$= 0 \leftarrow \boxed{r}$$

Since $0<1$, by the ratio test $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{\delta^n}$ converges

By the Weierstrass M-test the series $\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{z^n}$ converges uniformly (and absolutely) on D.

By the analytic convergence theorem

(a) $g(z) = \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$ is analytic on $A$,

and

(b) if $z \in A$, then

$$g'(z) = \sum_{n=1}^{\infty} \left( \frac{1}{n!} \frac{1}{z^n} \right)'$$

$$= \sum_{n=1}^{\infty} \frac{-n}{n!} \cdot \frac{1}{z^{n+1}}$$

$$\boxed{\begin{aligned} &\left(z^{-n}\right)' \\ &= -n z^{-n-1} \end{aligned}}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \frac{1}{z^{n+1}}$$