Math 5680

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Theorem: Let $A \subseteq \mathbb{C}$ be an open set.

① Suppose $f_n : A \to \mathbb{C}$ for $n \geq 1$ and $f : A \to \mathbb{C}$.
Suppose f_n is continuous on A for all $n \geq 1$.
If f_n converges uniformly to f on A, then f is continuous on A.

② Consequently, if functions $g_k(z)$ are continuous on A and
$$g(z) = \sum_{k=1}^{\infty} g_k(z)$$
converges uniformly on A, then $g(z)$ is continuous on A.

Proof:

① Let $z_0 \in A$.
We will show that f is continuous at z_0.

Let $\varepsilon > 0$.

Since $f_n \longrightarrow f$ uniformly on A, there exists $N > 0$ where

$$|f_N(z) - f(z)| < \varepsilon/3$$

for all $z \in A$.

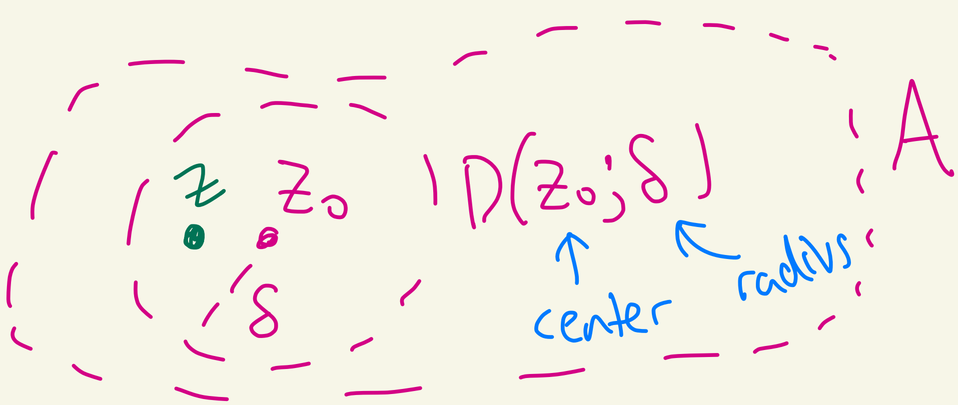
[So, f_N approximates f on A with error at most $\varepsilon/3$]

def cont.

$$\lim_{z \to z_0} f_N(z) = f_N(z_0)$$

Since f_N is continuous at z_0 there exists $\delta > 0$ where if $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$

Since A is open shrink δ so $D(z_0;\delta) \subseteq A$



So, if $|z - z_0| < \delta$, then

$z \in A$ and

$$|f(z) - f(z_0)| =$$

$$= |f(z) - f_N(z) + f_N(z) - f_N(z_0) + f_N(z_0) - f(z_0)|$$

$$\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

So, f is continuous at z_0.

(2) We are given that $g_k(z)$ are each continuous on A.

Then,

$$s_n(z) = \sum_{k=1}^{n} g_k(z)$$

are continuous on A for each $n \geq 1$.

Our sequence of functions on A is

$$\left.\begin{aligned} s_1(z) &= g_1(z) \\ s_2(z) &= g_1(z) + g_2(z) \\ s_3(z) &= g_1(z) + g_2(z) + g_3(z) \\ &\vdots \end{aligned}\right\} \text{The } s_n \text{ are the } f_n \text{ from (1)}$$

We are also assuming that

$$s_n \longrightarrow g \text{ uniformly on } A$$

where $g(z) = \sum_{k=1}^{\infty} g_k(z)$.

By ① g is continuous on A. $\blacksquare$

Theorem (Cauchy criterion)

Let $A \subseteq \mathbb{C}$.

① Let $f_n : A \to \mathbb{C}$ for $n \geq 1$.
Then, f_n converges uniformly on A iff for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$ then

$$|f_n(z) - f_{n+p}(z)| < \varepsilon$$

for all $z \in A$ and $p \geq 1$.

[$n+p$ is taking the place of m in the usual Cauchy def]

(2) Let $g_k : A \to \mathbb{C}$ for $k \geq 1$.

Then the series $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on A iff for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$ then

$$\underbrace{\left|\sum_{k=n+1}^{n+p} g_k(z)\right|}_{\underbrace{\left|\sum_{k=1}^{n+p} g_k(z) - \sum_{k=1}^{n} g_k(z)\right|}_{\left|S_{n+p}(z) - S_n(z)\right|}} < \varepsilon$$

for all $z \in A$ and $p \geq 1$

<u>proof:</u>

(1) ($\Rightarrow$) Suppose (f_n) converges uniformly on A.

Then there exists $f : A \to \mathbb{C}$ that (f_n) converges uniformly to.

Let $\varepsilon > 0$.

Then there exists $N > 0$ where if $n \geq N$ then

$$|f_n(z) - f(z)| < \varepsilon/2$$

for all $z \in A$.

Thus, if $n \geq N$ and $p \geq 1$ and $z \in A$ then

$$|f_n(z) - f_{n+p}(z)|$$

$$= |f_n(z) - f(z) + f(z) - f_{n+p}(z)|$$

$$\overset{\triangle}{\leq} |f_n(z) - f(z)| + |f(z) - f_{n+p}(z)|$$

$n \geq N$
$n+p \geq N$

$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon.$$

($\Leftarrow$) We are assuming "for every $\varepsilon > 0$, there is an $N > 0$ where if $n \geq N$ then $|f_n(z) - f_{n+p}(z)| < \varepsilon$ for all $z \in A$ and $p \geq 1$"

This implies that for each $z \in A$ the sequence $(f_n(z))$ is a Cauchy sequence.

Thus for each $z \in A$ we may define $f(z) = \lim_{n \to \infty} f_n(z)$.

That is $f_n \to f$ pointwise on A.

Let's show $f_n \to f$ uniformly on A.

Let $\varepsilon > 0$.

By our assumption there is an $N > 0$ where if $n \geq N$ then

$$|f_n(z) - f_{n+p}(z)| < \varepsilon/2$$

for all $z \in A$ and $p \geq 1$.

For each $z \in A$ pick p_z large enough so that

$$|f_{n+p_z}(z) - f(z)| < \varepsilon/2$$

for all $n \geq 1$.

using $f_n \to f$ point-wise

Thus if $n \geq N$ and $z \in A$, then

$$|f_n(z) - f(z)|$$

$$= |f_n(z) - f_{n+p_z}(z) + f_{n+p_z}(z) - f(z)|$$

$$\overset{\triangle}{\leq} |f_n(z) - f_{n+p_z}(z)| + |f_{n+p_z}(z) - f(z)|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

Thus, $f_n \to f$ uniformly on A.

② Apply part 1 to

$$S_n(z) = \sum_{k=1}^{n} g_k(z).$$

Then you'll get that

$$\sum_{k=1}^{\infty} g_k(z) \text{ converges}$$

uniformly on A iff for every $\varepsilon > 0$ there is an $N > 0$ where if $n \geq N$ then

$$\underbrace{\left| s_n(z) - s_{n+p}(z) \right|}_{\left| \sum_{k=1}^{n} g_k(z) - \sum_{k=1}^{n+p} g_k(z) \right| = \left| \sum_{k=n+1}^{n+p} g_k(z) \right|} < \varepsilon$$

for all $z \in A$, $p \geq 1$.

$\blacksquare$