Math 5680

3/20/23

HW 3

#6 Should say "simple" in problem statement.

Should say "simple, closed" in solutions

I fixed this online

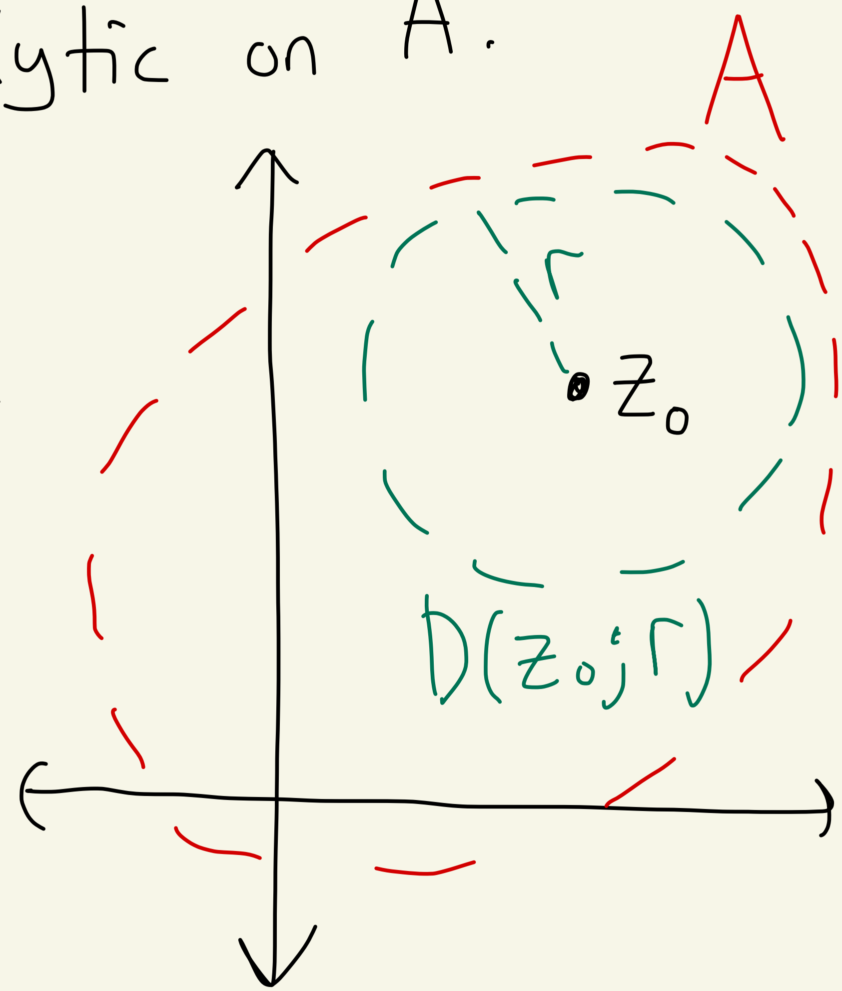
Tests handed back on Weds

(topic 3 continued...)

Discussion of the zeros of an analytic function

Suppose that $f : A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is an open set and f is analytic on A.

Let $z_0 \in A$ where $f(z_0) = 0$.



Since A is open there exists $r > 0$ where

$D(z_0;r) \subseteq A$

By Taylor's theorem

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$= \frac{f^{(1)}(z_0)}{1!}(z-z_0)^1 + \frac{f^{(2)}(z_0)}{2!}(z-z_0)^2 + \dots$$

$f^{(0)}(z_0) = f(z_0) = 0$

for all $z \in D(z_0; r)$.

<u>Case 1: Suppose $f^{(k)}(z_0) = 0$ for all $k \geq 0$</u>

Then, $f(z) = 0$ for all $z \in D(z_0; r)$

<u>Case 2: There exists a smallest positive integer n where $f^{(n)}(z_0) \neq 0$</u>

Then for $z \in D(z_0; r)$ we have

$$f(z) = \overbrace{\frac{f^{(n)}(z_0)}{n!}}^{\text{first non-zero term}}(z-z_0)^n + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0)^{n+1} + \cdots$$

$$= (z-z_0)^n\left[\underbrace{\frac{f^{(n)}(z_0)}{n!}}_{\text{not zero}} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0) + \cdots\right]$$

$$= (z-z_0)^n\varphi(z)$$

where $\varphi(z) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(n+k)}(z_0)}{(n+k)!}(z-z_0)^k$

and φ converges on $D(z_0; r)$

and $\varphi(z_0) = \dfrac{f^{(n)}(z_0)}{n!} \neq 0$.

φ is analytic on $D(z_0; r)$

since its a power series.

Summary of Case 2:

$$f(z) = (z - z_0)^n \varphi(z)$$

where φ is analytic at z_0

and $\varphi(z_0) \neq 0$

In this case, we say that f has a zero of order n at z_0.

Ex: $\boxed{f(z) = e^{(z-1)^2} - 1 , \quad z_0 = 1}$

$$f(z_0) = f(1) = e^{(1-1)^2} - 1 = e^0 - 1$$
$$= 1 - 1 = 0$$

For any $z \in \mathbb{C}$ we have

$$\boxed{e^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \quad \forall w \in \mathbb{C}}$$

$$f(z) = e^{(z-1)^2} - 1$$

$$= -1 + \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^{2n}$$

$$= \cancel{-1} + \left[\cancel{1} + (z-1)^2 + \frac{(z-1)^4}{2!} + \frac{(z-1)^6}{3!} + \ldots \right]$$

$$= (z-1)^2 \left[1 + \frac{1}{2!}(z-1)^2 + \frac{1}{3!}(z-1)^4 + \ldots \right]$$

$\varphi(z)$

$$= (z-1)^2 \varphi(z)$$

φ is analytic at $z_0 = 1$
and $\varphi(z_0) = \varphi(1) = 1 \neq 0$

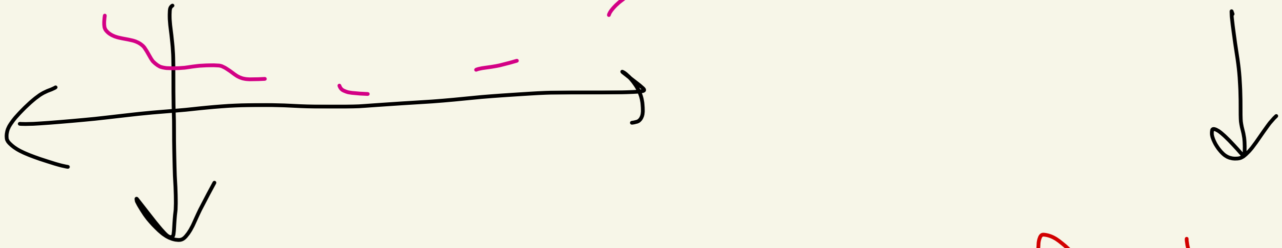
$z_0 = 1$ is a zero of order 2
for $f(z)$.

Theorem: (Isolation of zeros of an analytic function)

Suppose that $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is open and $f(z_0) = 0$ for some $z_0 \in A$. Suppose f is analytic on A.

Then either:

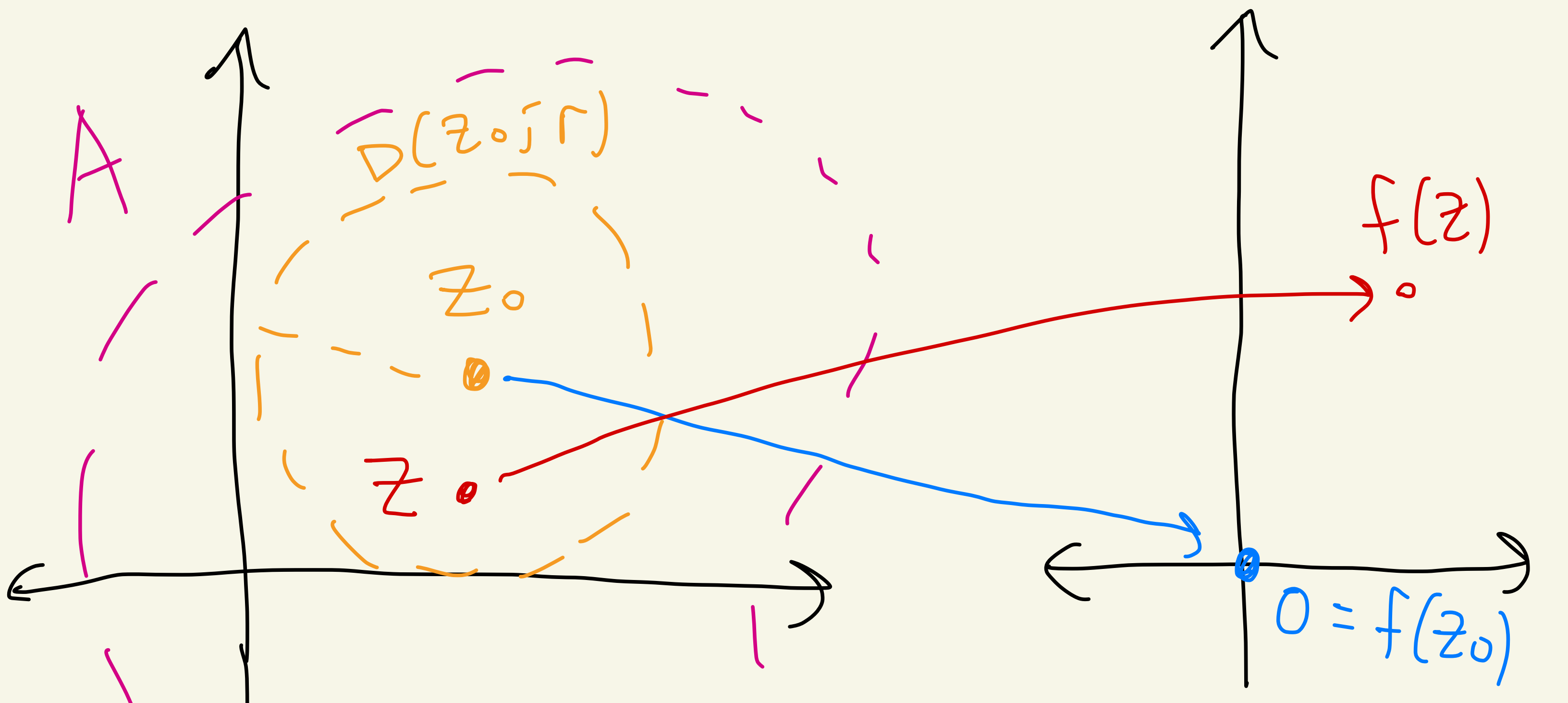
① There exists $r > 0$ where $D(z_0; r) \subseteq A$ and $f(z) = 0$ for all $z \in D(z_0; r)$



[f is locally the zero function at z_0]

OR

2) there exists $r > 0$ such that $D(z_0; r) \subseteq A$ and $f(z) \neq 0$ for all $z \in D(z_0; r) - \{z_0\}$



$(z \neq z_0)$

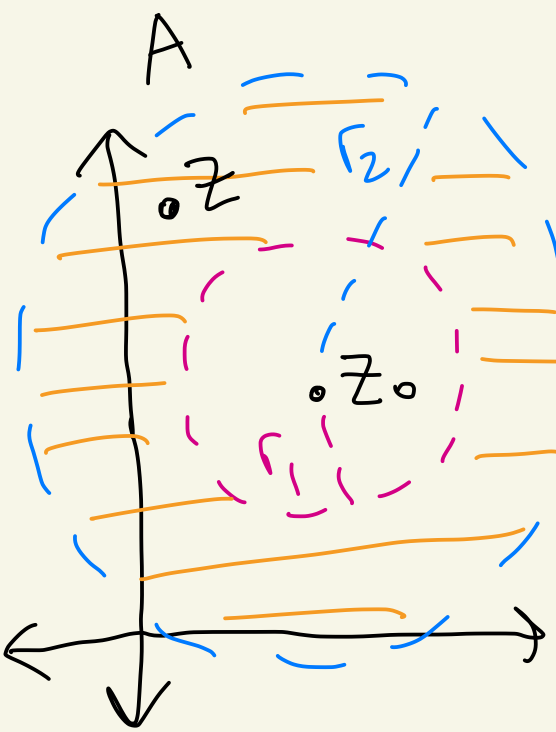
proof: HW 3 #7 $\blacksquare$

TOPIC 4 - Laurent Series

Theorem (Laurent Expansion Theorem)

Let $0 \leq r_1 < r_2$ and $z_0 \in \mathbb{C}$

Consider the annulus



$$A = \{ z \mid r_1 < |z - z_0| < r_2 \}$$

We allow $r_1 = 0$ and/or $r_2 = \infty$

Suppose $f : A \to \mathbb{C}$ is analytic on A.

Then we can write

$$\boxed{f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n}$$

$$= \left[\cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \right]$$

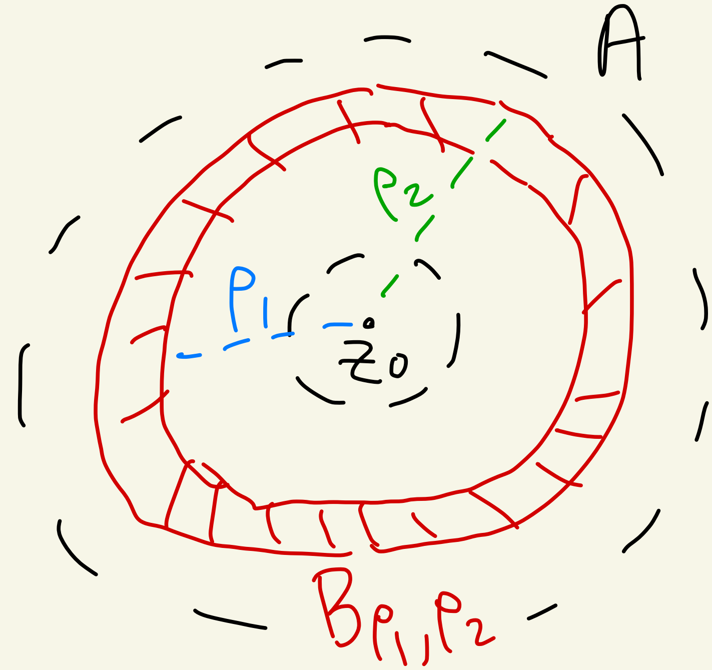
for $z \in A$.

Both series above converge absolutely on A and uniformly in sets of the form

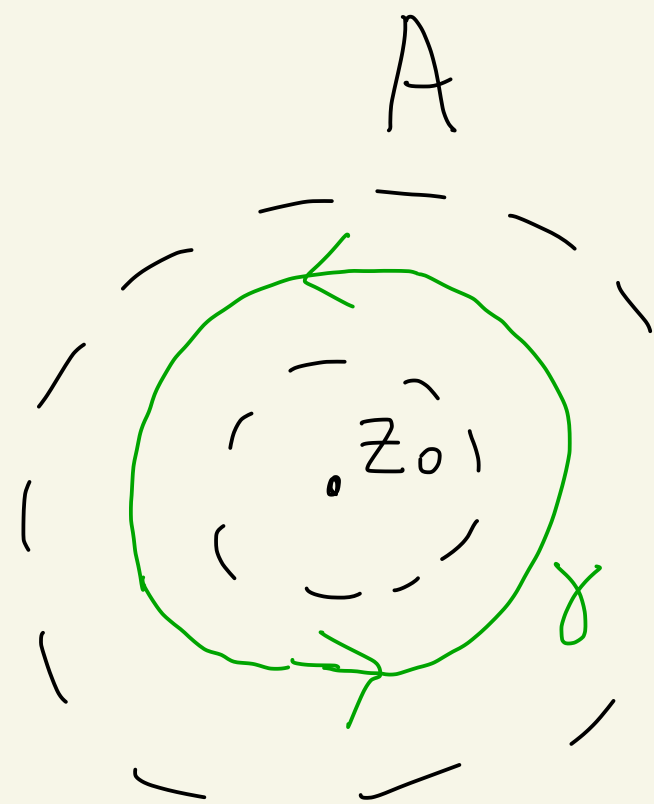
$$B_{\rho_1,\rho_2} = \{ z \mid \rho_1 \leq |z-z_0| \leq \rho_2 \}$$

where $r_1 < \rho_1 < \rho_2 < r_2$.

This series for f is called the <u>Laurent series of f centered at z_0 in the annulus A</u>.



If γ is a circle around z_0, oriented counter-clockwise, with radius r where $r_1 < r < r_2$ then



$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

for $n = 0, 1, 2, 3, \ldots$

and

$$b_n = \frac{1}{2\pi i} \int_{\gamma} f(w) \cdot (w - z_0)^{n-1}\, dw$$

for $n = 1, 2, 3, \ldots$

The Laurent series for f is unique. That is, any pointwise convergent expansion of f of this form in A equals the Laurent expansion.

<u>proof:</u> Hoffman/Marsden book $\blacksquare$