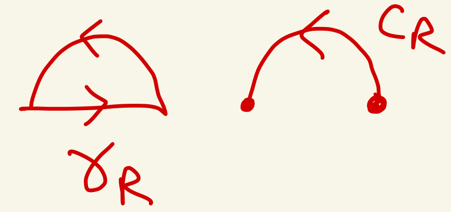
Math 5680

4/17/23

Recall from last week:



$\boxed{R > 1}$

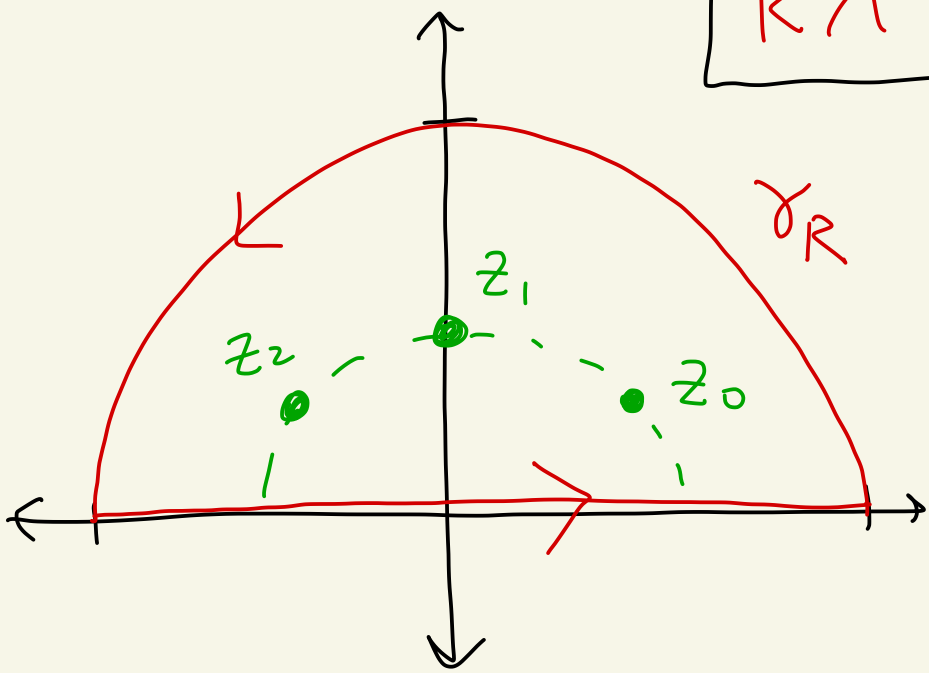
Want $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}\,dx$

$$f(z) = \frac{z^2}{z^6+1}$$

$$z_0 = e^{i\pi/6}$$

$$z_1 = e^{i3\pi/6}$$

$$z_2 = e^{i5\pi/6}$$



$$\int_{\gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx$$

and also

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \sum_{k=1}^{3} \operatorname{Res}(f; z_k)$$

Let's calculate the residues.

z_0, z_1, z_2 are all simple poles.

why?

$$f(z) = \frac{z^2}{z^6+1} = \frac{g(z)}{h(z)}$$

$$z_0 = e^{\frac{\pi}{6}i}$$

$$z_1 = e^{\frac{3\pi}{6}i}$$

$$z_2 = e^{\frac{5\pi}{6}i}$$

$$g(z_k) = z_k^2 \neq 0$$

$$h(z_k) = 0$$

$$h'(z_k) = 6z_k^5 \neq 0$$

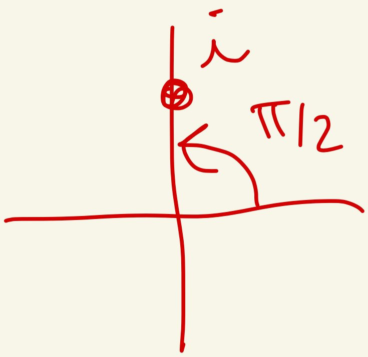
So, we have simple poles

and $\operatorname{Res}(f; z_k) = \dfrac{g(z_k)}{h'(z_k)} =$

$$= \frac{z_k^2}{6 z_k^5} = \frac{1}{6}\cdot\frac{1}{z_k^3}$$

So,

$$\text{Res}(f; z_0) = \frac{1}{6}\left(\frac{1}{e^{\pi i/6}}\right)^3 = \frac{1}{6}\left(\frac{1}{e^{3\pi i/6}}\right)$$



$$= \frac{1}{6}\left(\frac{1}{e^{\pi i/2}}\right) = \frac{1}{6}\left(\frac{1}{i}\right)$$

$$= \frac{1}{6}(-i) = \boxed{\frac{-i}{6}}$$

$$\text{Res}(f; z_1) = \frac{1}{6}\,\frac{1}{z_1^3} = \frac{1}{6}\left(\frac{1}{e^{3\pi i/6}}\right)^3$$

$$= \frac{1}{6}\left(\frac{1}{i}\right)^3 = \frac{1}{6}\left(\frac{1}{-i}\right)$$

$$= (1/6)(i) = \boxed{i/6}$$

$$\operatorname{Res}(f;z_2) = \frac{1}{6}\frac{1}{z_2^3} = \frac{1}{6}\left(\frac{1}{e^{5\pi i/6}}\right)^3$$

$$\boxed{\frac{15\pi}{6} = 2\pi + \frac{\pi}{2}}$$

$$= \frac{1}{6}\left(\frac{1}{e^{15\pi i/6}}\right) = \frac{1}{6}\left(\frac{1}{e^{\pi i/2}}\right)$$

$$= \frac{1}{6}\left(\frac{1}{i}\right) = \boxed{-\frac{i}{6}}$$

Thus,

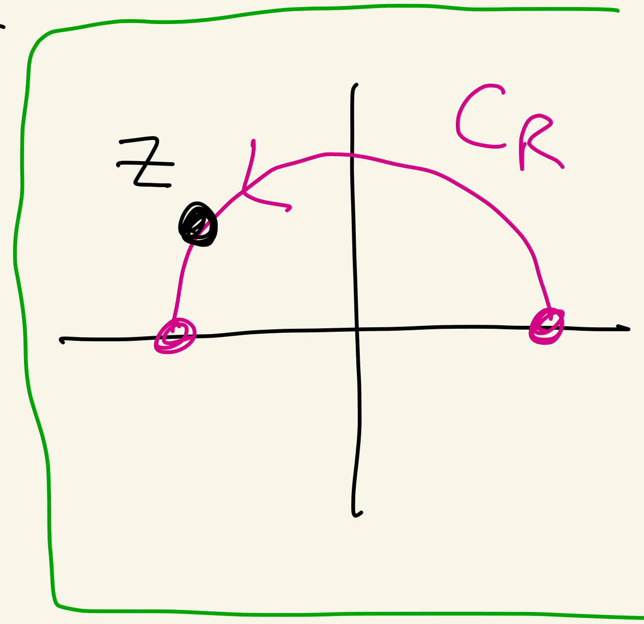
$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \underbrace{2\pi i\left[-\frac{i}{6}+\frac{i}{6}-\frac{i}{6}\right]}_{\pi/3}$$

So,

$$\boxed{\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} - \int_{C_R}\frac{z^2}{z^6+1}\,dz}$$

Let's let $R \to \infty$.

Let's show $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^6+1}\,dz = 0$



Let $z \in C_R$.

Then, $|z| = R$.

Thus,

$$|z^6+1| \geq \big||z^6| - |1|\big| = \big||z|^6 - 1\big|$$
$$= |R^6 - 1|$$
$$= R^6 - 1$$

4680, $a, b \in \mathbb{C}$
$|a+b| \geq \big||a| - |b|\big|$

$R > 1$
$R^6 - 1 > 0$

Thus, for $z \in C_R$ we have

$$\left|\frac{z^2}{z^6+1}\right| = \frac{|z|^2}{|z^6+1|} \leq \frac{R^2}{R^6-1}$$

$$|z|^2 = R^2$$
$$|z^6+1| \geq R^6-1$$

Therefore,

$$\left|\int_{C_R} \frac{z^2}{z^6+1}\,dz\right| \leq \underbrace{\frac{R^2}{R^6-1}}_{\text{bound on } f \text{ on } C_R} \cdot \underbrace{\pi R}_{\text{length of } C_R}$$

$$= \frac{\pi R^3}{R^6-1} = \frac{\pi\left(\frac{1}{R^3}\right)}{1-\frac{1}{R^3}} \longrightarrow$$

$$\longrightarrow \frac{\pi(0)}{1-0} = 0 \text{ as } R \to \infty$$

Thus,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx = \frac{\pi}{3} - \underbrace{\lim_{R\to\infty}\int_{C_R}\frac{z^2}{z^6+1}\,dz}_{0}$$

So,

$$\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}\,dx = \frac{\pi}{3}$$

Since f is an even function

$$\int_{0}^{\infty}\frac{x^2}{x^6+1}\,dx = \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$$

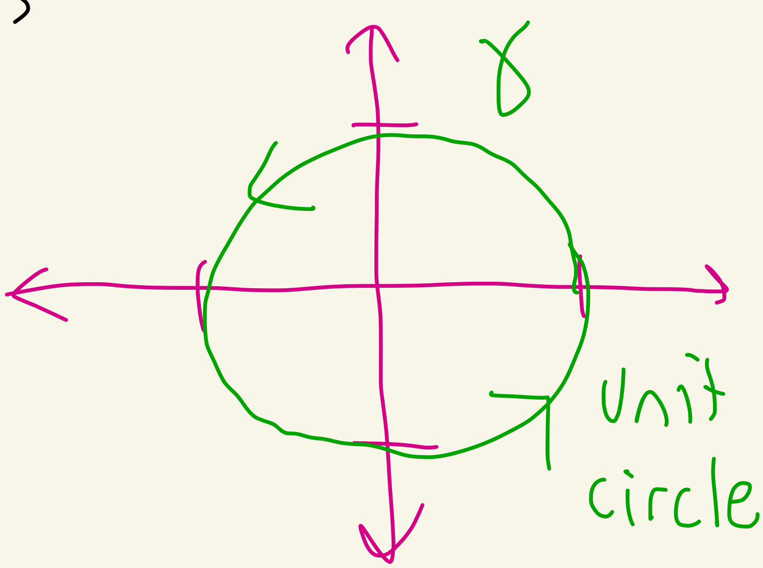
Application I – Definite integrals involving sine and cosine

<u>Ex:</u> We will calculate

$$\int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)}$$

Let $z = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

Let γ be the curve traced out by this equation.



$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$|e^{i\theta}| = 1$

Thus,

$$z = e^{i\theta}$$
$$dz = ie^{i\theta}\,d\theta$$
$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\boxed{\cos(w) = \frac{e^{iw}+e^{-iw}}{2} \qquad \sin(w) = \frac{e^{iw}-e^{-iw}}{2i}}$$

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z+\frac{1}{z}}{2}$$

So,

$$\int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)} = \int_\gamma \frac{\frac{1}{iz}\,dz}{5-4\left(\frac{z+1/z}{2}\right)}$$

$\boxed{\text{Why?}}$ $$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(\theta))\cdot\gamma'(\theta)\,d\theta$$

$$\int_\gamma \frac{\frac{1}{iz}\,dz}{5-4\left(\frac{z+\frac{1}{z}}{2}\right)} = \int_0^{2\pi} \frac{\cancel{\frac{1}{ie^{i\theta}}}}{5-4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}\cdot \cancel{ie^{i\theta}}\,d\theta$$

$$\boxed{\begin{aligned}&\gamma(\theta) = e^{i\theta},\ 0 \le \theta \le 2\pi\\ &\gamma'(\theta) = ie^{i\theta}\end{aligned}}$$

$$= \int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)}$$

We have

$$\int_\gamma \frac{\frac{1}{iz}\,dz}{5-4\left(\frac{z+1/z}{2}\right)} = \frac{1}{i}\int_\gamma \frac{1}{5-2z-\frac{2}{z}}\cdot\frac{1}{z}\,dz$$

$$= \overset{-i}{\frac{1}{i}}\int_\gamma \frac{dz}{-2z^2+5z-2}$$

$$= i\int_\gamma \frac{dz}{2z^2-5z+2}$$

When is $2z^2-5z+2=0$?

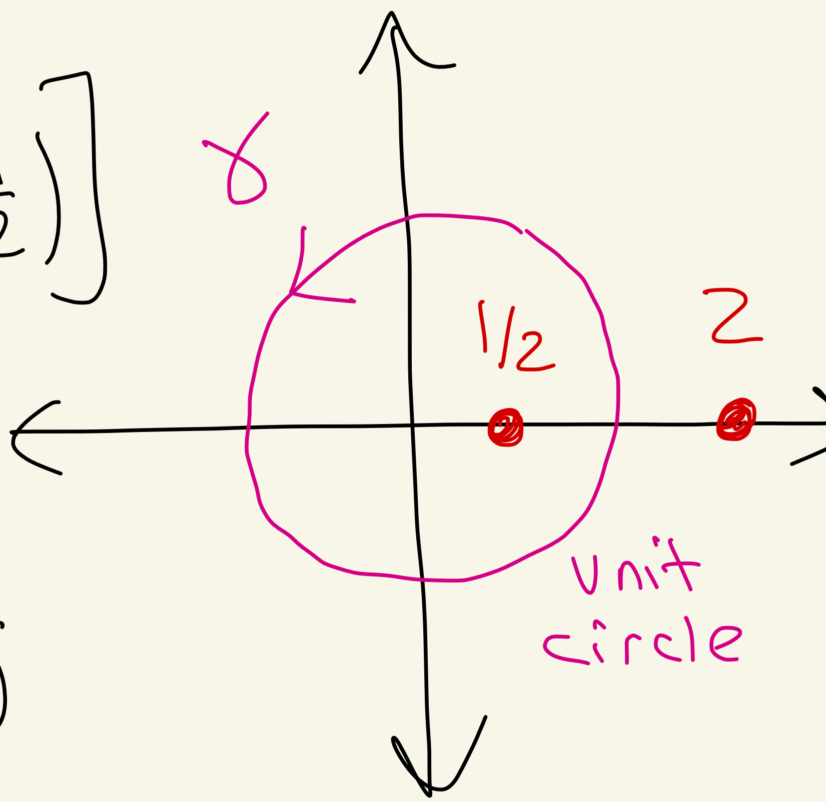
When $z = \dfrac{-(-5)\pm\sqrt{(-5)^2-4(2)(2)}}{2(2)}$

$$= \frac{5 \pm \sqrt{9}}{4} = 2, \frac{1}{2}$$

Thus,

$$i \int_\gamma \frac{dz}{2z^2 - 5z + 2} = i \int_\gamma \frac{dz}{2(z-2)(z-\frac{1}{2})}$$

$$= i \left[2\pi i \operatorname{Res}\left(f; \tfrac{1}{2}\right) \right]$$



where

$$f(z) = \frac{1}{2(z-2)(z-1/2)}$$

We have

$$\operatorname{Res}\left(f;\tfrac{1}{2}\right) = \varphi^{(1-1)}\left(\tfrac{1}{2}\right)\Big/(1-1)! = \varphi\left(\tfrac{1}{2}\right) = \frac{1}{2}\,\frac{1}{\left(\frac{1}{2}-2\right)} = -\frac{1}{3}$$

$$f(z) = \frac{\frac{1}{2}\cdot\frac{1}{(z-2)}}{(z-1/2)} = \frac{\varphi(z)}{z-1/2}$$

φ is analytic at $1/2$, $\varphi\left(\tfrac{1}{2}\right) \neq 0$

$1/2$ is a simple pole

Thus,

$$\int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)} = i\left[2\pi i \operatorname{Res}\left(f;\tfrac{1}{2}\right)\right]$$

$$= i\left[2\pi i\left(-\tfrac{1}{3}\right)\right]$$

$i^2 = -1$

$$= 2\pi/3$$

In general, suppose $R(x,y)$ is a rational function [ratio of polynomials] in x and y whose denominator does not vanish on the unit circle.

To evaluate

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta$$

make the substitution $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$ and use

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$d\theta = \frac{dz}{iz}$. Then use the residue theorem

One we had was $R(x,y) = \frac{1}{5-4x}$