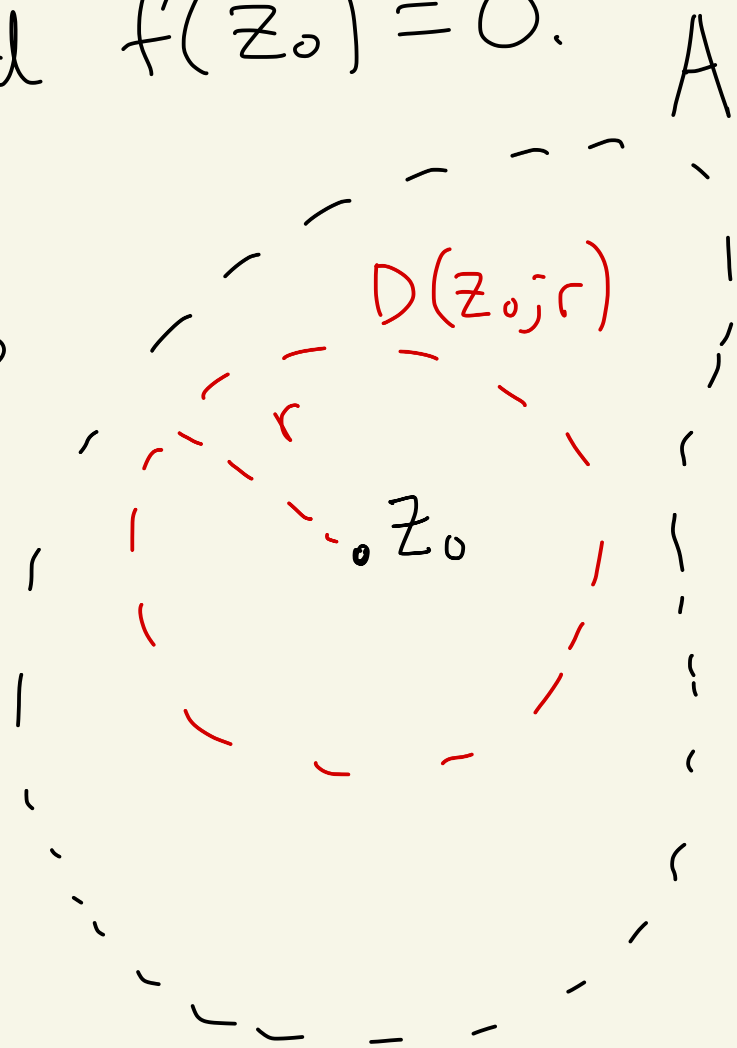
Math 5680

4/19/23

$\boxed{\text{HW 3}}$

7) Let $f : A \to \mathbb{C}$ be analytic on an open set A.

Let $z_0 \in A$ and $f(z_0) = 0$.



Since A is open there exists $r > 0$ where

$$D(z_0; r) \subseteq A$$

By Taylor's theorem

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

for all $z \in D(z_0; r)$

Case 1: Suppose $f^{(k)}(z_0) = 0$ for all $k \geq 0$.

Then, $f(z) = \sum_{k=0}^{\infty} \frac{0}{k!}(z - z_0)^k = 0$

for all $z \in D(z_0; r)$.

Case 2: otherwise, there exists a smallest $k \geq 1$ where $f^{(k)}(z_0) \neq 0$.

Then,

$$f(z) = \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k + \frac{f^{(k+1)}(z_0)}{(k+1)!}(z - z_0)^{k+1} + \dots$$

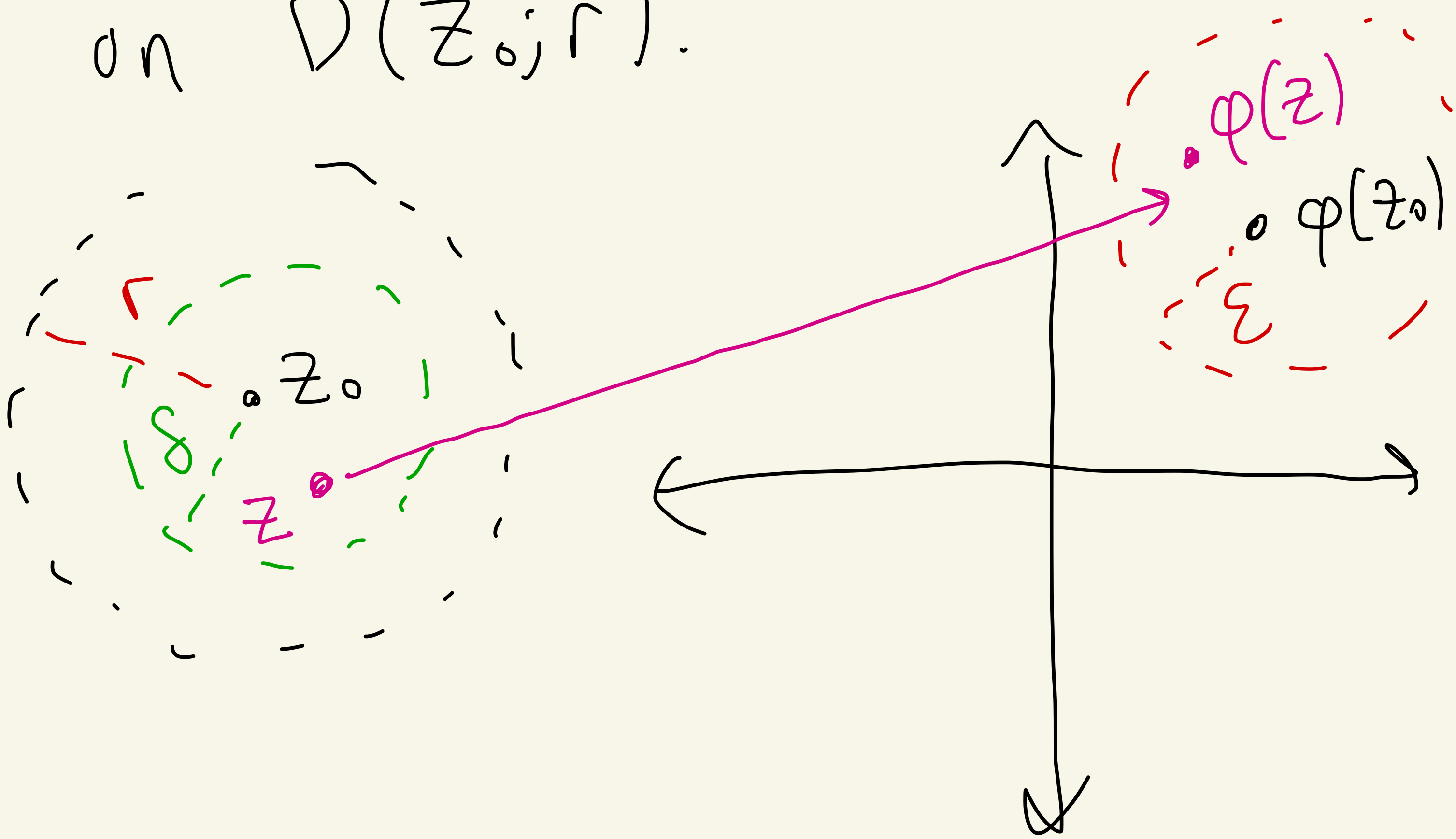
$$= (z - z_0)^k \underbrace{\left[\sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^{n-k} \right]}_{\varphi(z)}$$

$$= (z - z_0)^k \varphi(z)$$

φ is analytic in $D(z_0; r)$

and $\varphi(z_0) = \dfrac{f^{(k)}(z_0)}{k!} \neq 0.$

Since φ is analytic in $D(z_0; r)$ we know φ is continuous on $D(z_0; r)$.



Let $\varepsilon = \frac{|\varphi(z_0)|}{2}$.

Since φ is continuous at z_0 there exists $0 < \delta < r$ where if $z \in D(z_0; \delta)$ then $\varphi(z) \in D(\varphi(z_0); \varepsilon)$

Thus, if $z \in D(z_0; \delta) - \{z_0\}$ then

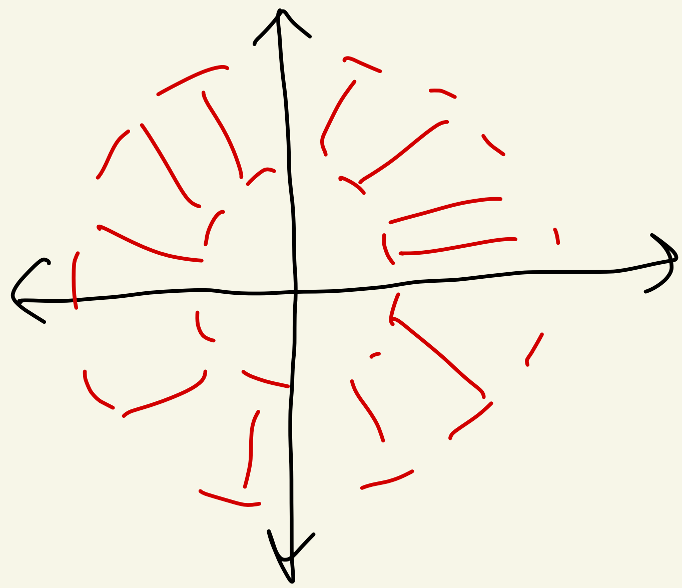
$$f(z) = \underbrace{(z - z_0)^k}_{\substack{\neq 0 \\ z \neq z_0}} \underbrace{\varphi(z)}_{\substack{\neq 0 \\ \varphi(z) \in D(\varphi(z_0); \varepsilon)}} \neq 0.$$

$\blacksquare$

HW 4 – Part 1

$\boxed{4(b)}$ $B = \{ z \mid 1 < |z| < 2 \}$

$$f(z) = \frac{1}{z(z-1)(z-2)}$$



$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1)$$

$$z = 0 \rightarrow \quad 1 = A(2) \rightarrow A = 1/2$$

$$z = 1 \rightarrow \quad 1 = B(1)(-1) \rightarrow B = -1$$

$$z = 2 \to 1 = C(2)(1) \to C = 1/2$$

$$\frac{1}{z(z-1)(z-2)} = \frac{1/2}{z} + \frac{-1}{z-1} + \frac{1/2}{z-2}$$

$$\boxed{1<|z|<2}$$

$$= \frac{1/2}{z} - \frac{1}{z}\underbrace{\left[\frac{1}{1-1/z}\right]}_{\left|\frac{1}{z}\right|<1 \text{ since } 1<|z|} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\underbrace{\left[\frac{1}{1-\frac{z}{2}}\right]}_{\left|\frac{z}{2}\right|<1 \text{ since } |z|<2}$$

$$= \frac{1/2}{z} - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$- \frac{1}{4}\left[1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots\right]$$

$$= \boxed{\frac{1/2}{z} - \frac{1}{z}} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots$$

$$- \frac{1}{2^2} - \frac{z}{2^3} - \frac{z^2}{2^4} - \frac{z^3}{2^5} - \cdots$$

$$= \left[\cdots - \frac{1}{z^4} - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1/2}{z} \right]$$

$$+ \left[-\frac{1}{2^2} - \frac{z}{2^3} - \frac{z^2}{2^4} - \frac{z^3}{2^5} - \cdots \right]$$

HW 4 - Part 2

1 (e) $f(z) = \dfrac{e^{z^2}}{(z-1)^4}$, $z_0 = 1$

We have

$$f(z) = \frac{\varphi(z)}{(z-1)^4}$$

where $\varphi(z) = e^{z^2}$, $\varphi(1) = e^{1^2} \neq 0$,

φ is analytic at $z_0 = 1$.

f has a pole of order 4 at $z_0 = 1$.

And

$$\operatorname{Res}(f;1) = \frac{\varphi^{(4-1)}(1)}{(4-1)!} = \frac{\varphi^{(3)}(1)}{6}$$

$$\varphi(z) = e^{z^2}$$

$$\varphi'(z) = 2ze^{z^2}$$

$$\varphi''(z) = 2e^{z^2} + 2z\left(2ze^{z^2}\right)$$

$$= 2e^{z^2} + 4z^2e^{z^2}$$

$$\varphi'''(z) = 2(2z)e^{z^2} + 8ze^{z^2} + 4z^2\left(2ze^{z^2}\right)$$

$$= 4ze^{z^2} + 8ze^{z^2} + 8z^3e^{z^2}$$

$$\operatorname{Res}(f;1) = \frac{\varphi'''(1)}{6} = \frac{4e+8e+8e}{6}$$

$$= \frac{20}{6}e = \frac{10}{3}e$$

Ex:

$$f(z) = \frac{e^{2z} - e^{2}}{(z-1)^4}, \quad z_0 = 1$$

Let $g(z) = e^{2z} - e^{2}$

$g(1) = 0$ so can't use φ theorem.

But g is analytic at $z_0 = 1$.

Let's find its Taylor series.

$$\left.\begin{aligned} g(z) &= e^{2z} - e^{2} \\ g'(z) &= 2e^{2z} \\ g''(z) &= 2^2 e^{2z} \\ g'''(z) &= 2^3 e^{2z} \end{aligned}\right\} \quad \begin{aligned} g^{(0)}(1) &= 0 \\ g^{(k)}(1) &= 2^k e^2 \\ k &\geq 1 \end{aligned}$$

g is analytic on all of $\mathbb{C}$.

So,

$$g(z) = \sum_{k=1}^{\infty} \frac{g^{(k)}(1)}{k!}(z-1)^k$$

$$= \sum_{k=1}^{\infty} \frac{2^k e^2}{k!}(z-1)^k$$

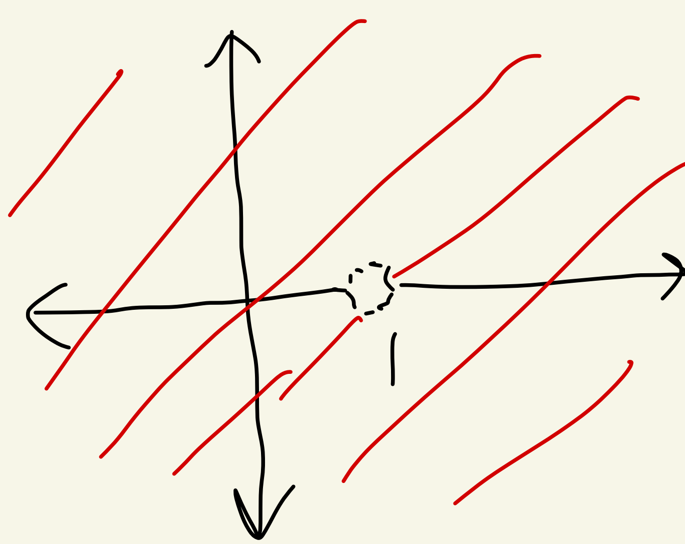
for all $z \in \mathbb{C}$.

Taylor!

Thus, if $z \neq 1$, then

$$f(z) = \frac{g(z)}{(z-1)^4}$$

$$= \frac{1}{(z-1)^4}\left[\sum_{k=1}^{\infty} \frac{2^k e^2}{k!}(z-1)^k\right]$$



$$= \frac{1}{(z-1)^4}\left[\frac{2e^2}{1!}(z-1) + \frac{2^2e^2}{2!}(z-1)^2 + \frac{2^3e^2}{3!}(z-1)^3 + \frac{2^4e^2}{4!}(z-1)^4 + \cdots\right]$$

pole

$$= \frac{2e^2}{(z-1)^3} + \frac{\left(\frac{2^2e^2}{2!}\right)}{(z-1)^2} + \frac{\frac{2^3e^2}{3!}}{(z-1)} + \frac{2^4e^2}{4!} + \frac{2^5e^2}{5!}(z-1) + \cdots$$

residue

f has a pole of order 3 at $z_0 = 1$

$$\mathrm{Res}(f;1) = \frac{2^3e^2}{3!} = \frac{2^2e^2}{3}$$