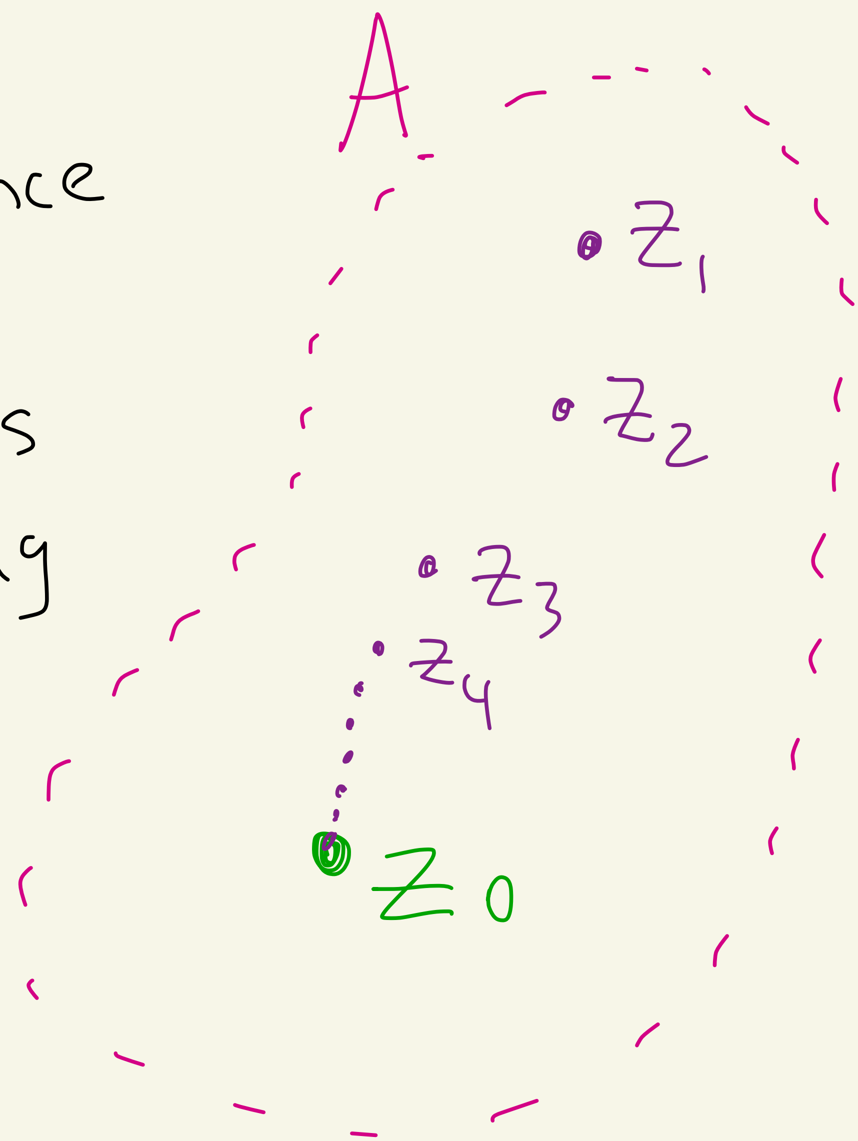
Math 5680

5/1/23

Identity Theorem: Let f and g be analytic in a region A. [region means open and path-connected]

Suppose there exists a sequence $z_1, z_2, z_3, \ldots$ of distinct points in A converging in A to z_0 in A, such that $f(z_n) = g(z_n)$ for $n = 1, 2, 3, \ldots$



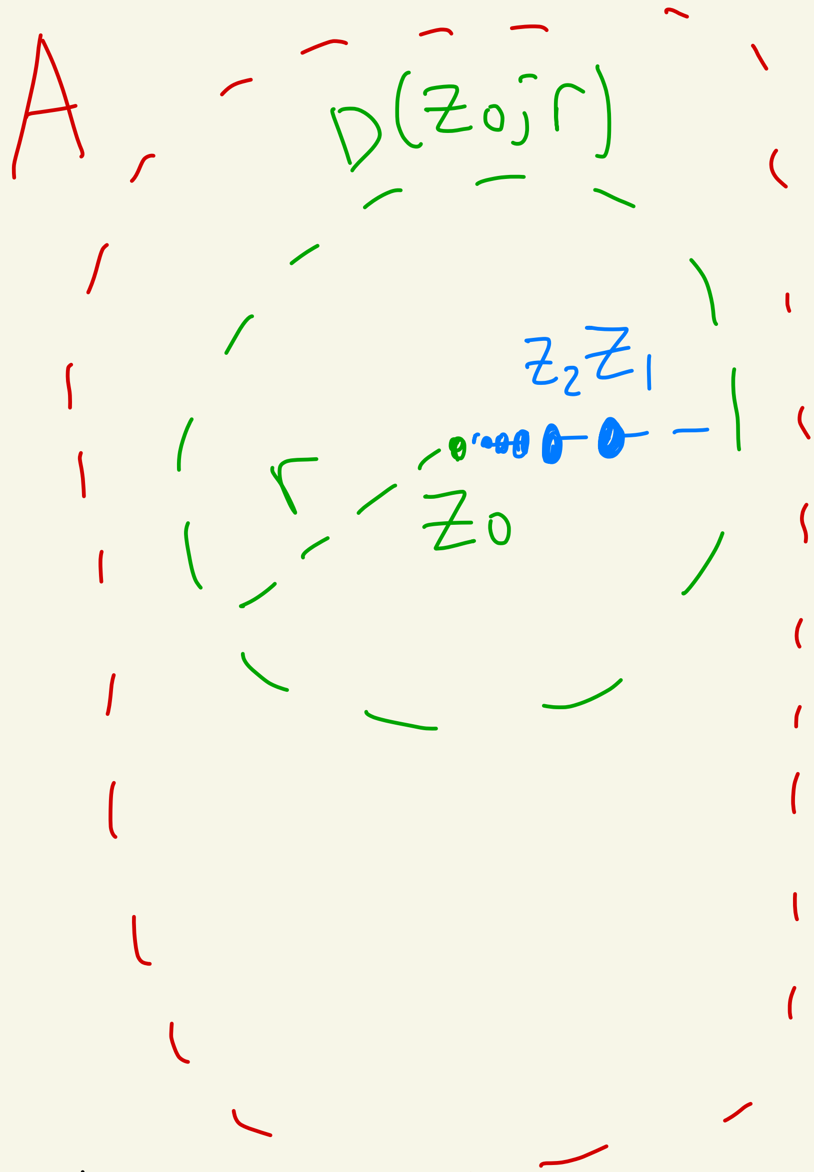
Then $f(z) = g(z)$ for all $z \in A$

We will prove later

Corollary: Let f and g be analytic in a region A. Suppose $f(z)=g(z)$ for all z in some open disc inside of A, then $f(z)=g(z)$ for all z in A.

proof:

Suppose $f(z)=g(z)$ for all z in $D(z_0;r) \subseteq A$.



Let
$$z_n = z_0 + \frac{r}{n+1}$$
for $n \geq 1$.

Then each z_n is inside of $D(z_0;r)$ and thus inside of A.

And $z_n \to z_0$ and $z_0 \in A$.

And $f(z_n) = g(z_n)$ for all $n \geq 1$ by assumption.

By the identity theorem $f(z) = g(z)$ for all $z \in A$.

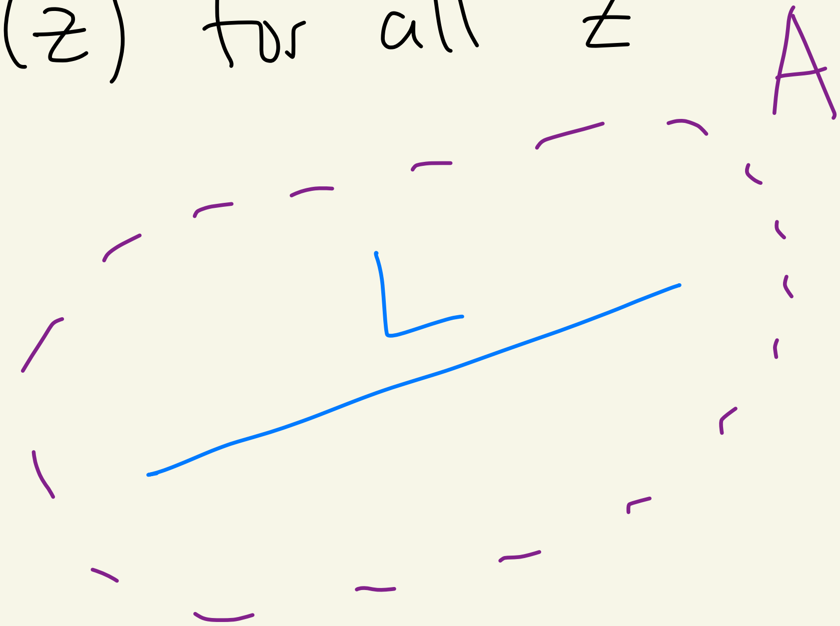
$\blacksquare$

<u>Corollary:</u> Let f and g be analytic in a region A.

Suppose there is a line segment L contained in A

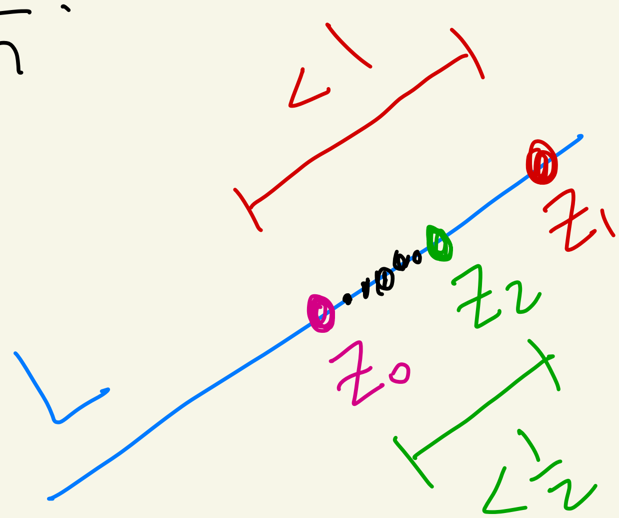
Suppose $f(z) = g(z)$ for all z on L.

Then $f(z)=g(z)$ for all z in A.



__proof:__

Let z_0 be at the middle of L. For each $n \geq 1$ pick a z_n on L where $|z_n - z_0| < \frac{1}{n}$.



Then, $z_n \in A$ for all $n \geq 0$ and $f(z_n) = g(z_n)$ for $n \geq 1$ and so by the identity thm $f(z)=g(z)$ for all $z \in A$

$\blacksquare$

Ex: Suppose $f: \mathbb{C} \to \mathbb{C}$ is an entire function. That is, f is analytic on all of $\mathbb{C}$.

Suppose also

$$f(x) = f(x+i0) = e^x$$

for all $x \in \mathbb{R}$.

[f is the real-valued e^x when restricted to the real line]

$$\boxed{\text{Claim: } f(z) = e^z \text{ for all } z \in \mathbb{C}}$$

Proof: Note that if $x \in \mathbb{R}$ and $z = x + 0i = x$, then

$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

$$e^z = e^{x+i0} = e^x(\underbrace{\cos(0)}_{1} + i\underbrace{\sin(0)}_{0})$$

$$= e^x = f(x) = f(z)$$

So $f(z)$ and e^z agree on the real-axis.

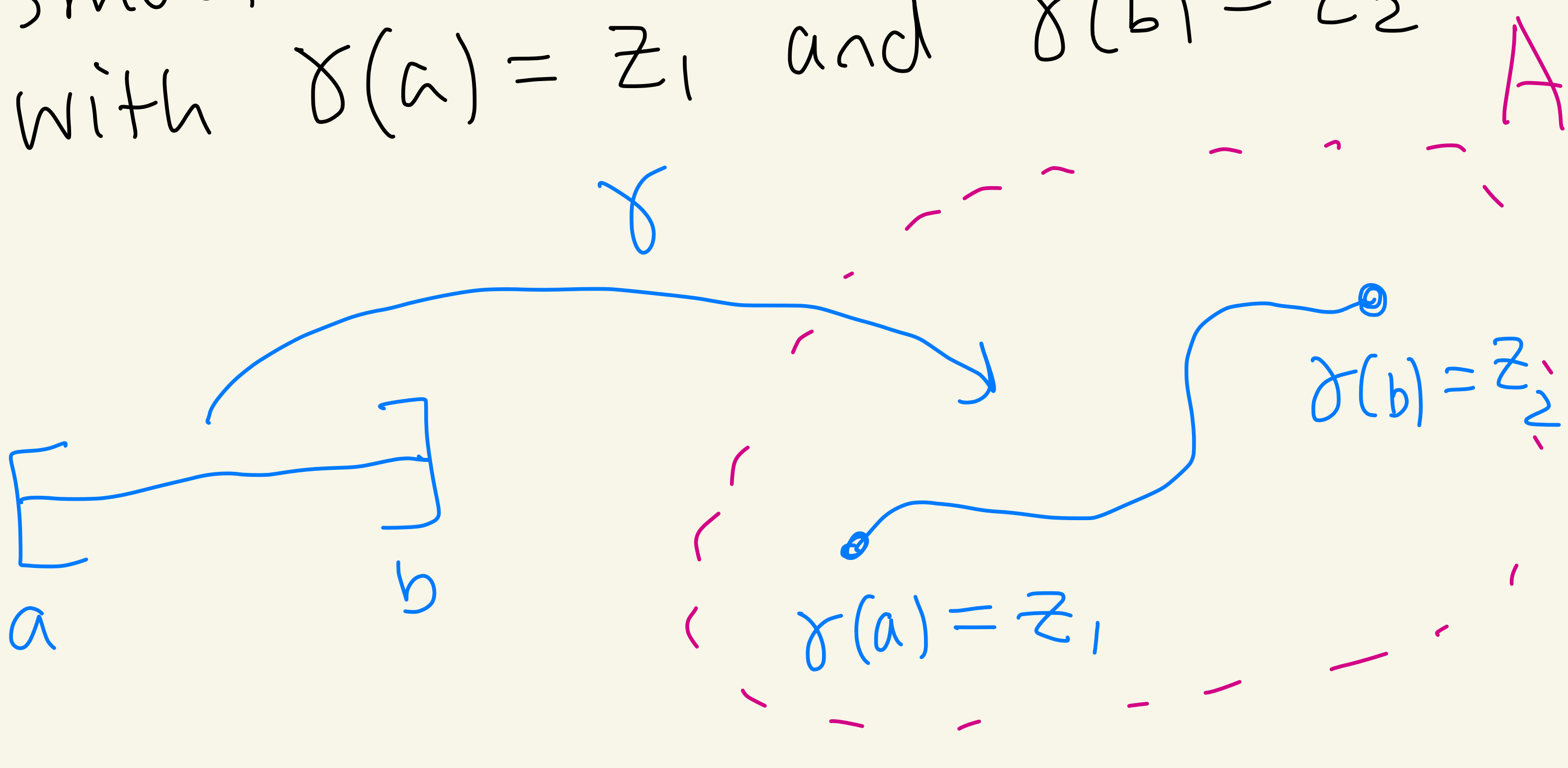
So, they agree on a line segment in $\mathbb{C}$.

Since f and e^z are both entire, by the identity thm corollary, $f(z) = e^z$ for all $z \in \mathbb{C}$. $\blacksquare$

You could apply the same reasoning to $\sin(z)$ and $\cos(z)$ for example.

In 4680:

$A \subseteq \mathbb{C}$ is path-connected if for every pair of points $z_1, z_2 \in A$ there exists a piece-wise smooth curve $\gamma : [a,b] \to A$ with $\gamma(a) = z_1$ and $\gamma(b) = z_2$



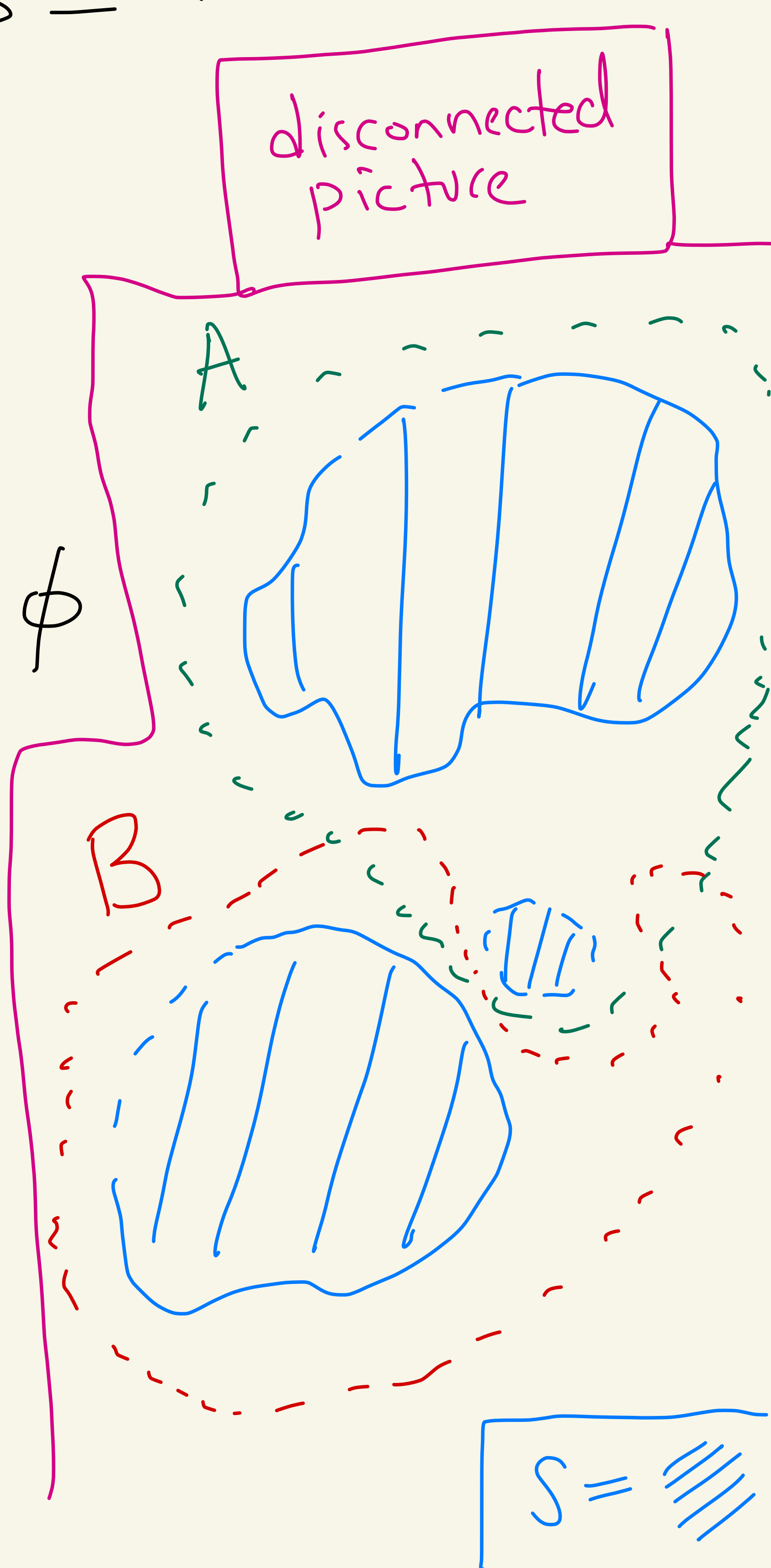
<u>Def:</u> A set $S \subseteq \mathbb{C}$ is <u>disconnected</u> if there exist open sets $A, B \subseteq \mathbb{C}$ such that:

① $S \subseteq A \cup B$

② $(S \cap A) \neq \phi$
$(S \cap B) \neq \phi$

③ $(S \cap A) \cap (S \cap B) = \phi$

If S is not disconnected then its called <u>connected</u>.



Theorem: Let $S \subseteq \mathbb{C}$ be an open set. Then S is connected iff S is path-connected.

proof: We will prove $(\Rightarrow)$. The proof of $(\Leftarrow)$ is in Hoffman / Marsden book.

$(\Rightarrow)$ Suppose S is open and connected.

We must show that S is path-connected.

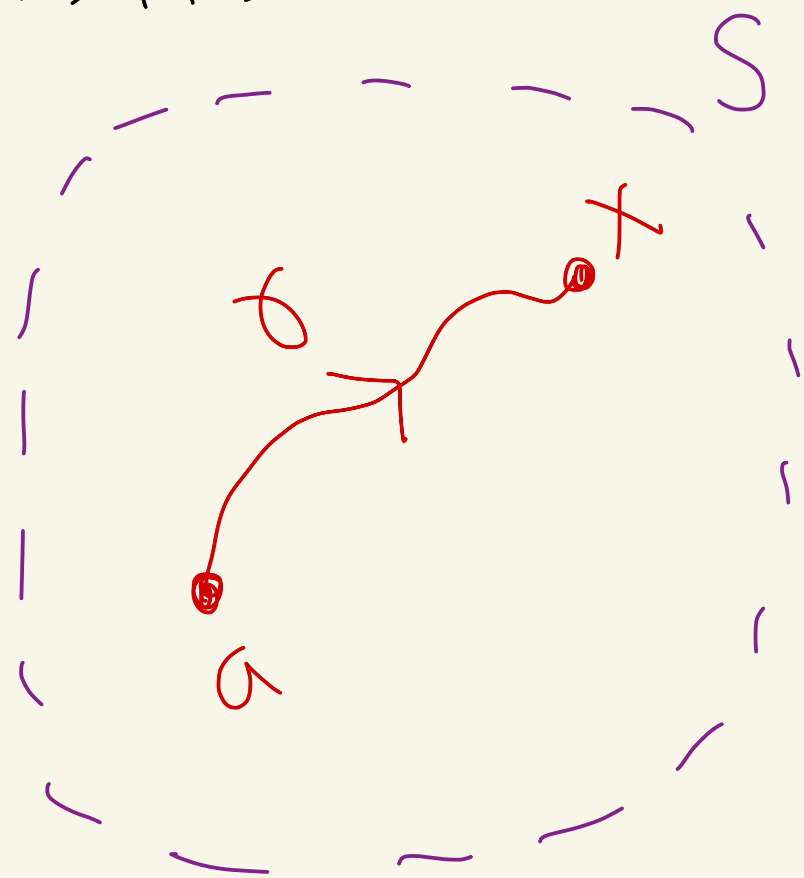
Fix some arbitrary $a \in S$

Let

$A = \{ x \in S \mid$ there exists a piecewise smooth curve γ starting at a and ending at x and γ lies in $S \}$

<u>Goal:</u> Show $A = S$.

This would show that S is path-connected since a is arbitrary.



picture of $x \in A$

Suppose to the contrary, that $A \neq S$.

Let $B = S - A = \{ x \mid x \in S, x \notin A \}$

We will show this implies S

is disconnected by A and B yielding a contradiction

① $S = A \cup (S-A) = A \cup B$

② $S \cap A \neq \phi$ because $a \in S \cap A$

$S \cap B \neq \phi$ because $B = S - A$ and we assumed $A \neq S$.

③ $(S \cap A) \cap (S \cap B) = A \cap B$

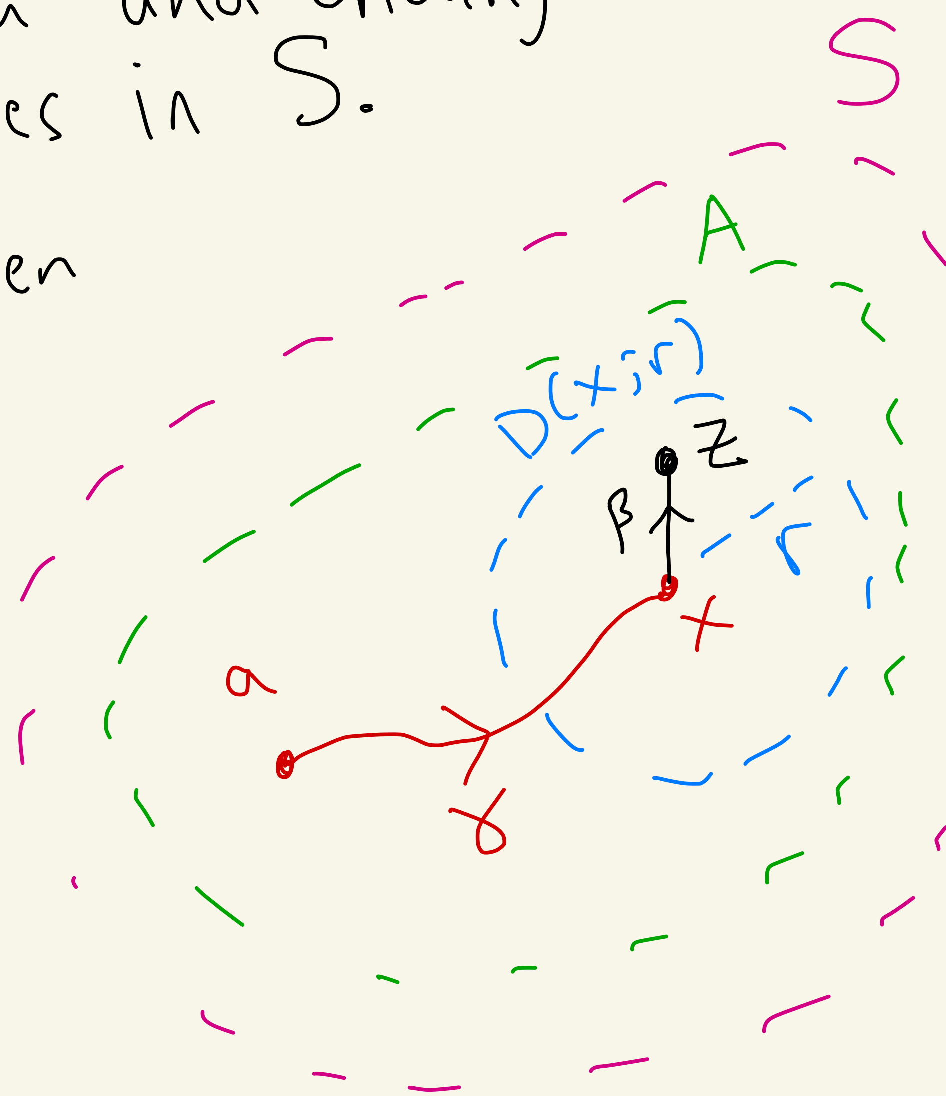
$= A \cap (S-A) = \phi$

④ We just need to show A and B are both open.

A is open: Let x be in A.

We need to show that x is an interior point of A.

Since x is in A there exists a piecewise smooth curve γ starting at a and ending at x where γ lies in S.



Since S is open there exists a disc $D(x;r) \subseteq S$ for some $r>0$.

Let's show in fact that $D(x;r) \subseteq A$ which makes A open.

Let $z \in D(x;r)$.

Let β be the straightline curve from x to z

Then β lies in $D(x;r) \subseteq S$.

Thus the curve $\gamma + \beta$ ← [γ first, then β]

lies in S and connects a to z.

Thus, $z \in A$.

So, $D(x;r) \subseteq A$.

Thus, x is an interior point and A is open.

(next time... B is also open...)