Math 5680

5/3/23

Summary of last time

<u>Theorem:</u> $S \subseteq \mathbb{C}$ is open.
S is connected iff S is path-connected.

<u>proof:</u> ($\Rightarrow$) Suppose S is open and connected.

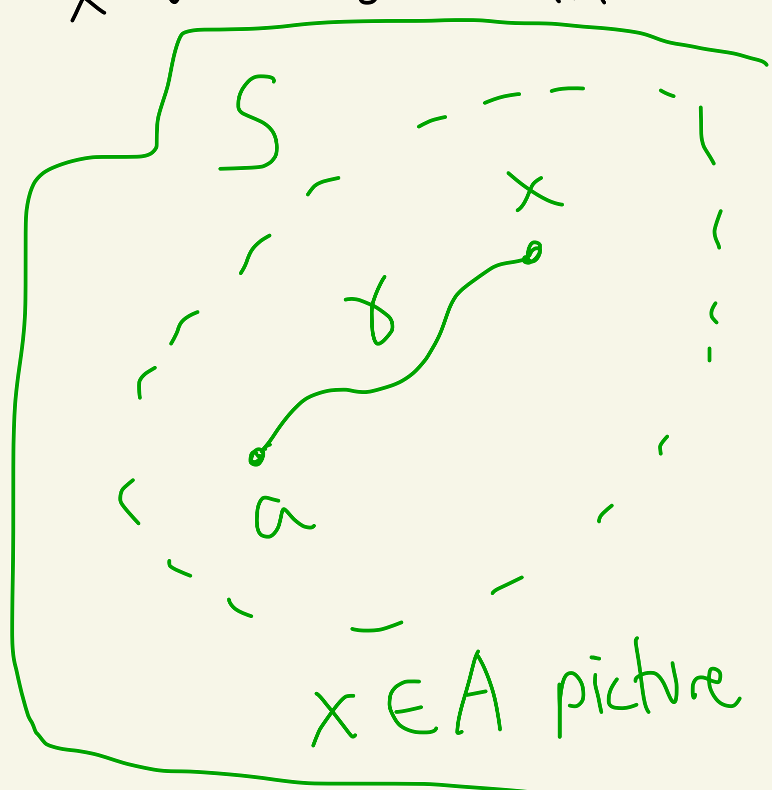
Fix $a \in S$.

Let

$$A = \left\{ x \in S \;\middle|\; \begin{array}{c} \text{there exists a piecewise} \\ \text{smooth curve } \gamma \text{ connecting} \\ a \text{ to } x \text{ where } \gamma \text{ lies in } S \end{array} \right\}$$

<u>Goal:</u> Show $A = S$.

We did a bunch of stuff.



$x \in A$ picture

All we had left was to show that
$B = S - A$ is open.

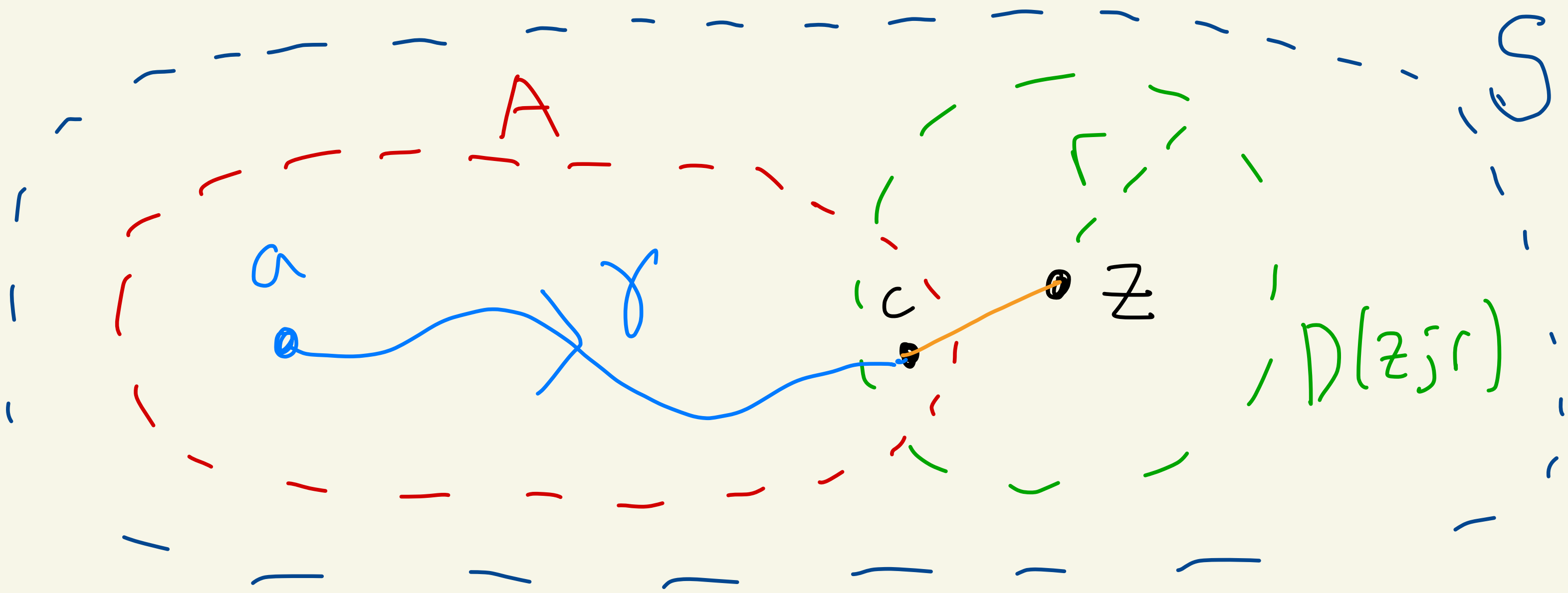
<u>$B = S - A$ is open</u>

Let $z \in B = S - A$.
We must show that z is an interior point of B.

Since $z \in S$ and S is open there exists $r > 0$ where
$D(z; r) \subseteq S$.

We will show this implies
$D(z; r) \subseteq B$ and we're done.

Suppose $D(z; r) \not\subseteq B$.



Then there exists $c \in A \cap D(z;r)$.

Since $c \in A$ there exists a path γ lying in A starting at a and ending at c.

Let β be the straight line starting at c and ending at z.

β lies in $D(z;r) \subseteq S$.

Then $\gamma + \beta$ connects a to z via a path through S.

That would imply $z \in A$, contradiction.

Thus, $D(z;r) \subseteq B$

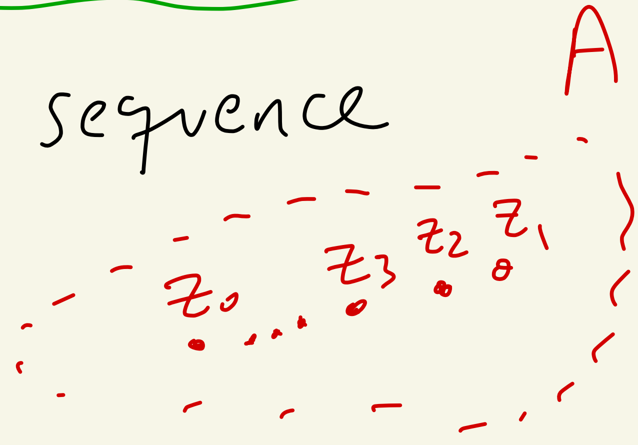
So, B is open.

Identity Theorem

Let f and g be analytic in a region A

open & path-connected/connected

Suppose there is a sequence of distinct points



$z_1, z_2, z_3, z_4, \ldots$

in A converging to $z_0 \in A$.

Suppose $f(z_n) = g(z_n)$

for $n \geq 1$.

Then $f(z) = g(z)$ for all $z \in A$.

<u>proof:</u> Let $h(z) = f(z) - g(z)$.

WTS $h(z) = 0 \ \forall z \in A$.

We know h is analytic on A

and $h(z_n) = 0 \ \forall n \geq 1$.

Since h is continuous on A,

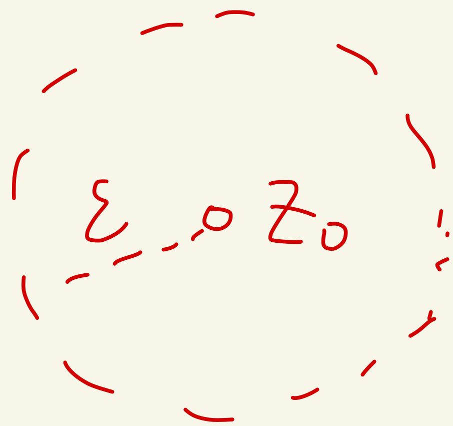
$$0 = \lim_{n \to \infty} h(z_n) \overset{\checkmark}{=} h\left(\lim_{n \to \infty} z_n\right) = h(z_0)$$

So, $h(z_0) = 0$.

So, z_0 is not an isolated zero of h.

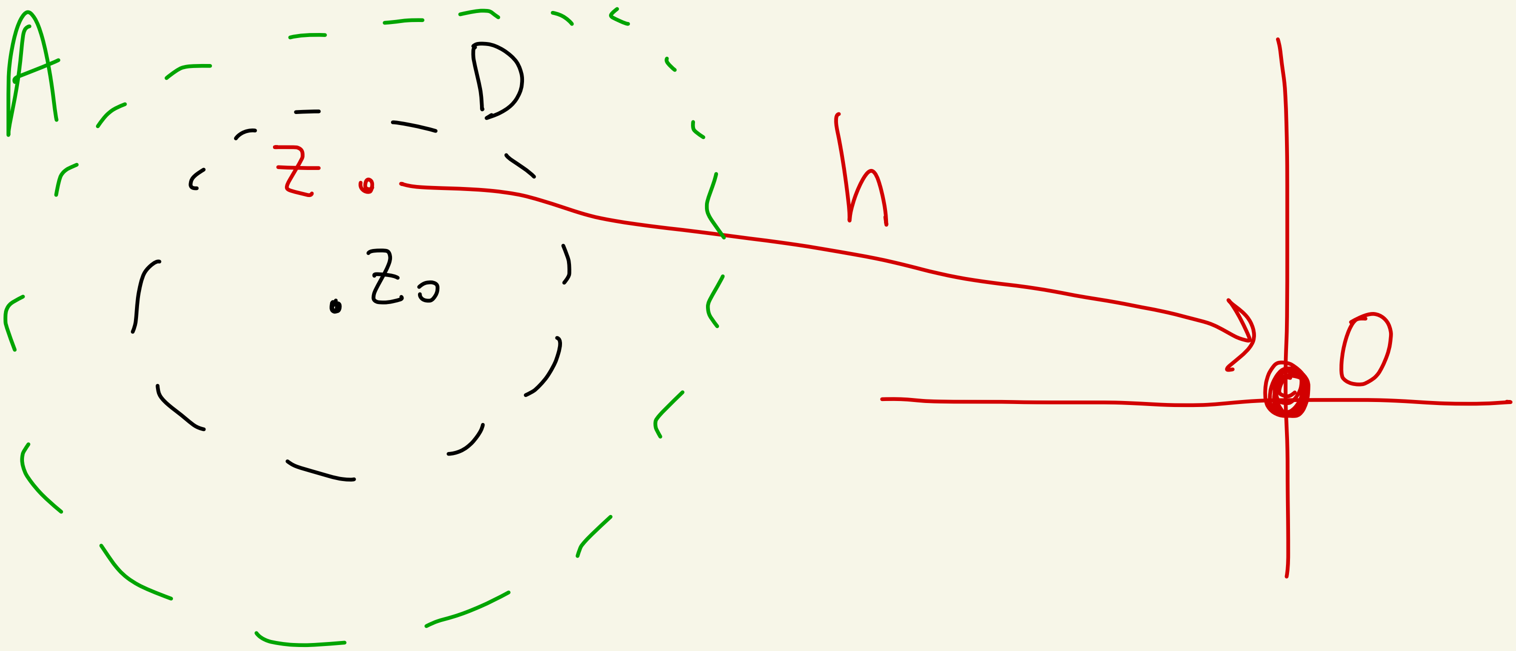
(Why?) Suppose z_0 is an isolated zero of h. Then $\exists \varepsilon > 0$ where $h(z) \neq 0$ $\forall z \in D^*(z_0; \varepsilon)$

But since $z_n \to z_0$ $\exists N$ where if $n \geq N$ then $\underbrace{z_n \in D(z_0; \varepsilon)}_{|z_n - z_0| < \varepsilon}$



Contradiction

By HW 3 #7, there must exist a disc $D \subseteq A$ centered at z_0 where $h(z)=0 \ \forall z \in D$.



Let

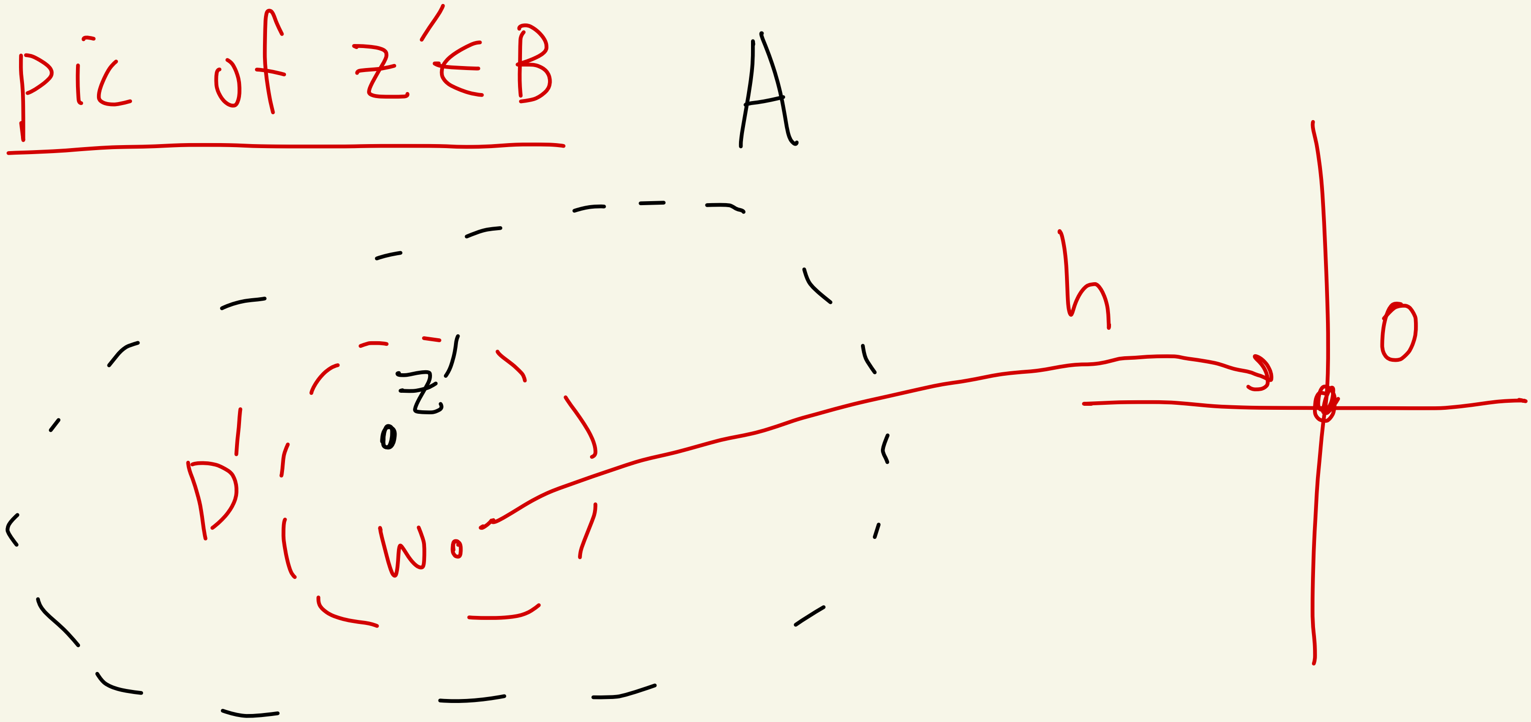
$$B = \{ z' \in A \mid \text{there exists a disc } D' \subseteq A \text{ with } z' \in D' \text{ and } h(w)=0 \ \forall w \in D' \}$$

We know $z_0 \in B$

using $D' = D$

from above.

pic of $z' \in B$



Goal: Show $B = A$

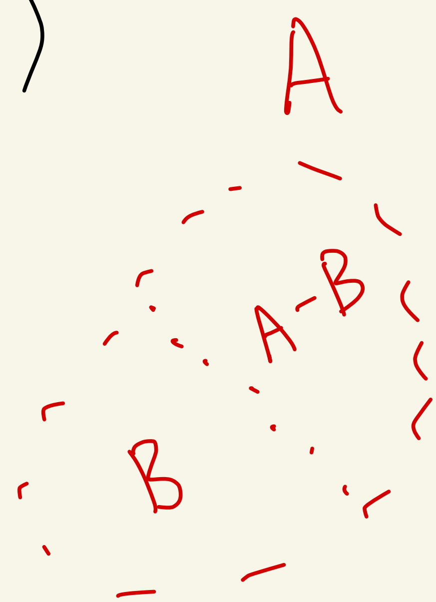
Suppose $B \neq A$.
We will show this leads to the contradiction that A is not connected.

Note $B \neq \emptyset$ because $z_0 \in B$.

Also, $A - B \neq \phi$ because we are assuming $B \neq A$.

Also, $A = B \cup (A - B)$

and $B \cap (A - B) = \phi$



If we show B and $A-B$ are both open then we have shown A is disconnected which is our contradiction.

<u>B is open:</u>

Let $z \in B$.

WTS z is an interior pt of B.

Since $z \in B$ there exists a disc $D(z;\rho) \subseteq A$ where $h(w) = 0 \quad \forall w \in D(z;\rho)$

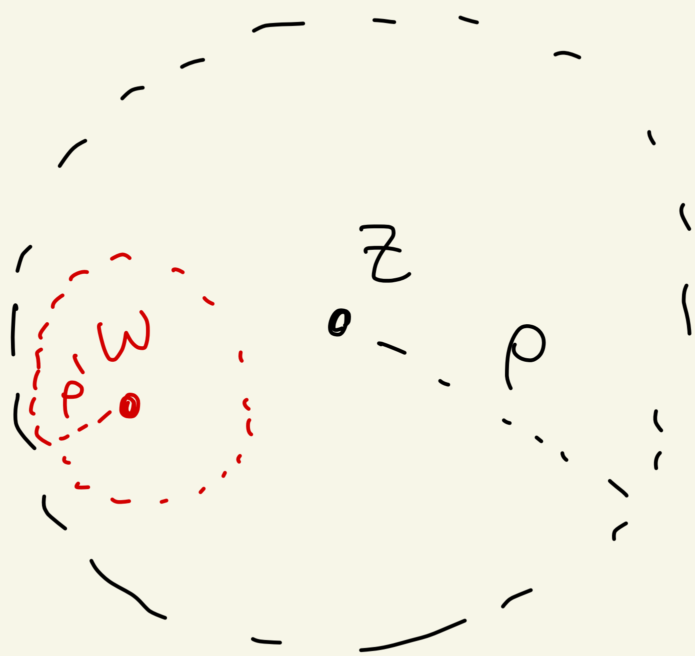
We want $D(z;\rho) \subseteq B$.

Pick $w \in D(z;\rho)$

Set $\rho' = \rho - |z-w|$

Then

$D(w;\rho') \subseteq D(z;\rho)$



Then, $D(w;\rho') \subseteq A$ and $h(w') = 0$ $\forall w' \in D(w;\rho')$.

So, $w \in B$.

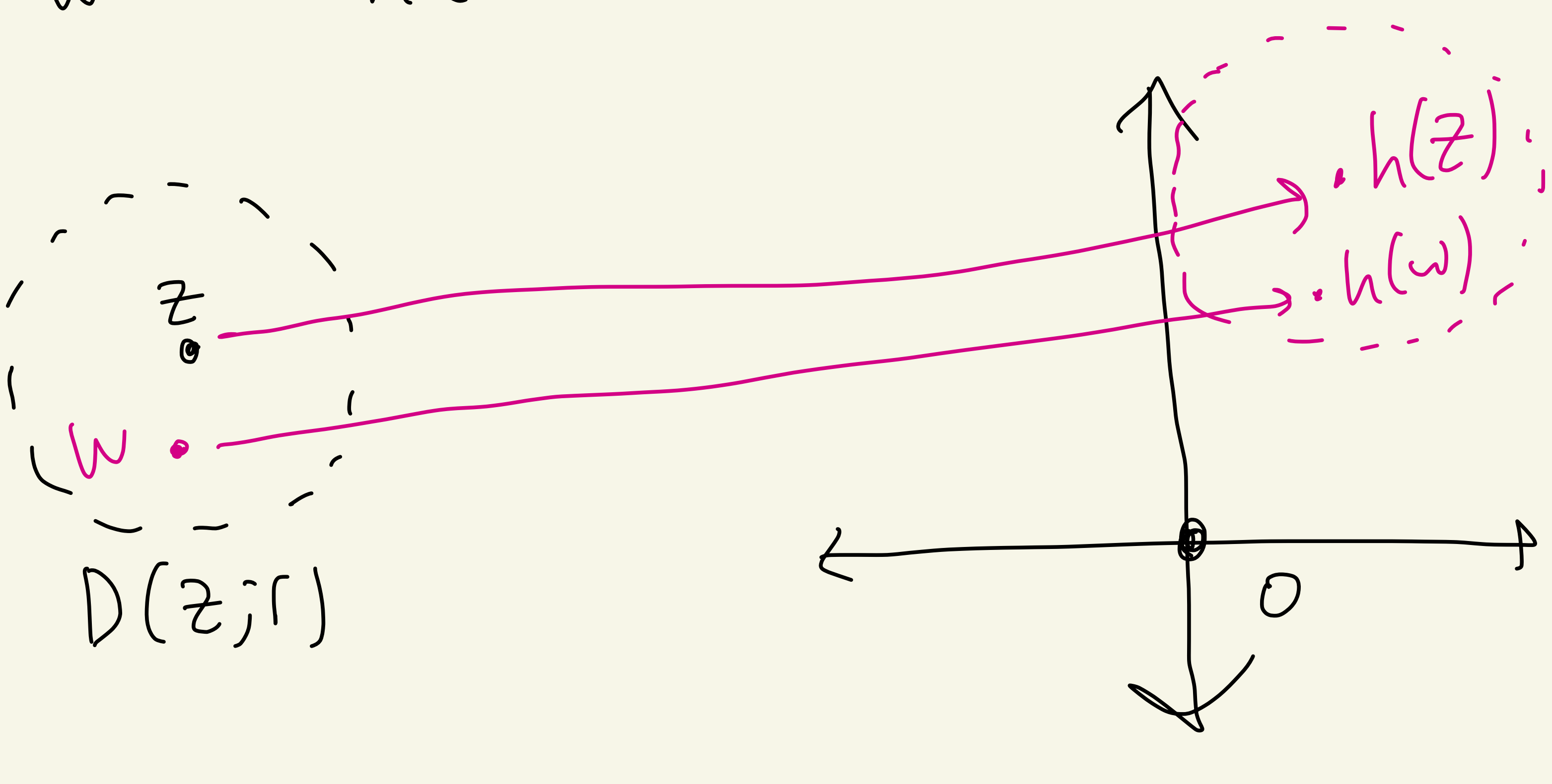
Thus, $D(z;\rho) \subseteq B$. And B is open.

$A - B$ is open

Let $z \in A - B$.
WTS z is an interior pt of $A-B$.

case 1: Suppose $h(z) \neq 0$

By 4680 HW 4 #5, since h is continuous at $z \in A$, there exists a disc $D(z;r) \subseteq A$ where $h(w) \neq 0 \ \forall w \in D(z;r)$



Thus, $D(z;r) \subseteq A - B$.

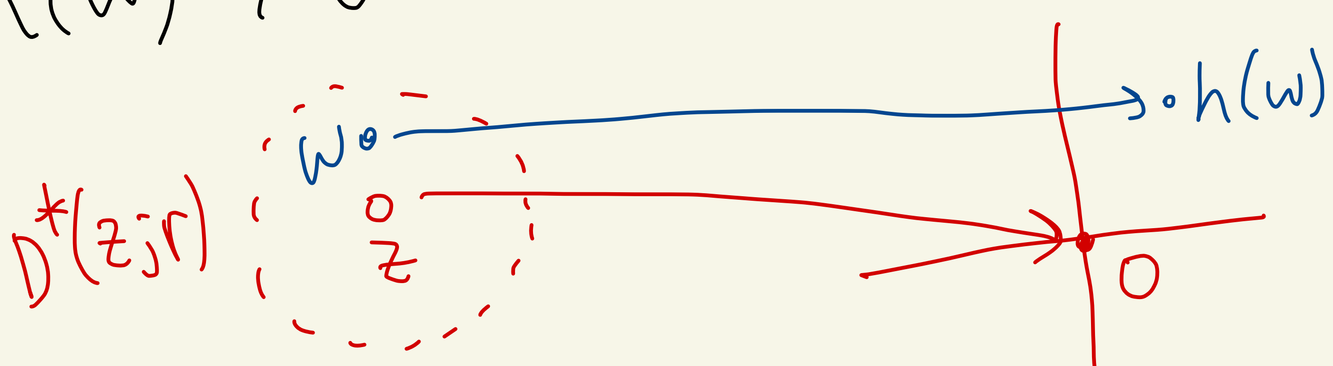
[Because if you pick $w \in D(z;r)$ there is no disc around w where the whole disc goes to 0.]

So, z is interior to $A - B$ and $A - B$ is open.

Case 2: Suppose $h(z) = 0$

Since $z \notin B$ this implies that z is an isolated zero of h.

By HW 3 #7, there is a disc $D(z;r) \subseteq A$ where $h(w) \neq 0 \quad \forall w \in D^*(z;r)$.



Thus, if $w \in D^*(z;r)$, then $w \notin B$.

So, $D(z;r) \subseteq A - B$.

So, z is an interior pt of $A - B$.

So, $A - B$ is open

Thus, $A - B$ is open by cases 1 & 2, making A disconnected.

Contradiction !

So $A = B$

And $h(z) = 0 \ \forall z \in A$. $\blacksquare$