5680

HW 1

Solutions

① (a)

$$\sum_{n=1}^{\infty} \frac{i^n}{2^{n-1}} = i \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^{n-1} = i\left[\left(\frac{i}{2}\right)^0 + \left(\frac{i}{2}\right)^1 + \left(\frac{i}{2}\right)^2 + \cdots\right]$$

$$= i \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$$

This is a geometric series.
Recall that $\sum_{n=0}^{\infty} z^n$ converges iff $|z|<1$, with sum equal to $\frac{1}{1-z}$ if it converges.

Here we have $z = \frac{i}{2}$ and $|z| = \left|\frac{i}{2}\right| = \frac{1}{2} < 1$.
Thus, $\sum_{n=1}^{\infty} \frac{i^n}{2^{n-1}}$ converges to the sum

$$\sum_{n=1}^{\infty} \frac{i^n}{2^{n-1}} = i \sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = i \frac{1}{1-\frac{i}{2}} = i \frac{2}{2-i}$$

$$= \boxed{\frac{2i}{2-i}}$$

① (b) We have that

$$\sum_{n=3}^{\infty} \frac{e+1}{2^n \pi^{n+3}} = (e+1) \sum_{n=3}^{\infty} \frac{1}{2^n \pi^{n+3}}$$

$$= (e+1) \left[\frac{1}{2^3 \cdot \pi^6} + \frac{1}{2^4 \cdot \pi^7} + \frac{1}{2^5 \cdot \pi^8} + \cdots \right]$$

$$= (e+1) \frac{1}{2^3 \cdot \pi^6} \left[1 + \frac{1}{2^1 \cdot \pi} + \frac{1}{2^2 \cdot \pi^2} + \cdots \right]$$

$$= (e+1) \frac{1}{2^3 \cdot \pi^6} \sum_{n=0}^{\infty} \frac{1}{2^n \pi^n} = (e+1) \frac{1}{2^3 \cdot \pi^6} \sum_{n=0}^{\infty} \left(\frac{1}{2 \cdot \pi} \right)^n$$

Note that $\left| \frac{1}{2\pi} \right| \approx 0.16 < 1$.

Thus, this geometric series converges to

$$(e+1) \frac{1}{2^3 \cdot \pi^6} \cdot \left(\frac{1}{1 - \frac{1}{2\pi}} \right) = \boxed{(e+1) \frac{1}{2^3 \cdot \pi^6} \left(\frac{2\pi}{2\pi - 1} \right)}$$

① (c)

Note that

$$\sum_{n=0}^{\infty} \frac{10^{n+1}}{2^n \sqrt{3}^{n+3}} = \frac{10^1}{2^0 \sqrt{3}^3} + \frac{10^2}{2^1 \cdot \sqrt{3}^4} + \frac{10^3}{2^2 \cdot \sqrt{3}^5} + \cdots$$

$$= \frac{10}{\sqrt{3}^3}\left[1 + \frac{10^1}{2^1 \cdot \sqrt{3}^1} + \frac{10^2}{\sqrt{2}^2 \cdot \sqrt{3}^2} + \cdots\right]$$

$$= \frac{10}{\sqrt{3}^3} \sum_{n=0}^{\infty} \frac{10^n}{2^n \cdot \sqrt{3}^n} = \frac{10}{\sqrt{3}^3} \sum_{n=0}^{\infty} \left(\frac{10}{2\sqrt{3}}\right)^n$$

Note that $\left|\frac{10}{2\sqrt{3}}\right| \approx 2.89 > 1.$

Thus, the above geometric series diverges.

① (d) Note that

$$\lim_{n\to\infty}\left(\frac{(1+i)^n}{5+(1+i)^n}\right)=\lim_{n\to\infty}\left(\frac{1}{\frac{5}{(1+i)^n}+1}\right)$$

divide top/bottom by $(1+i)^n$

Also,

$$\lim_{n\to\infty}\left|\frac{5}{(1+i)^n}\right|=\lim_{n\to\infty}\frac{|5|}{|(1+i)^n|}$$

$$=\lim_{n\to\infty}\frac{5}{|1+i|^n}=\lim_{n\to\infty}\frac{5}{\sqrt{2}^n}=0$$

Thus, $\lim_{n\to\infty}\frac{5}{(1+i)^n}=0.$

$\lim|a_n|=0$ iff $\lim a_n=0$

So, $\lim_{n\to\infty}\left[\frac{(1+i)^n}{5+(1+i)^n}\right]=\frac{1}{0+1}=1\neq 0.$

So, by the divergence theorem, $\sum_{n=1}^{\infty}\frac{(1+i)^n}{5+(1+i)^n}$ diverges.

1 (e)

We need to use partial fractions here.

Let's solve

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

which becomes

$$1 = A(n+1) + Bn \qquad (*)$$

This must be true for all n.

Plug in $n=-1$ into $(*)$ to get

$$1 = A(0) + B(-1)$$

$$B = -1$$

Plug in $n=0$ into $(*)$ to get

$$1 = A(1) + B(0)$$

$$A = 1.$$

Thus,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \text{for } n \geq 1.$$

Let's look at the partial sums

$$S_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left[\frac{1}{n} - \frac{1}{n+1} \right]$$

We have that

$$S_1 = \underbrace{\left(\frac{1}{1} - \frac{1}{2}\right)}_{n=1} = 1 - \frac{1}{2}$$

$$S_2 = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{1}{2}} - \frac{1}{3}\right)}_{n=2} = 1 - \frac{1}{3}$$

$$S_3 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$\vdots$$

In general

$$S_k = 1 - \frac{1}{k+1}$$

So,

$$\lim_{k\to\infty} \sum_{n=0}^{k} \frac{1}{n(n+1)} = \lim_{k\to\infty} S_k =$$

$$= \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)$$

$$= 1 - 0$$

$$= 1$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

② If $n_0 = 1$, the series are the same.
Suppose $n_0 > 1$.

Let $s_k = a_1 + \dots + a_k$ be the partial sums of $\sum_{n=1}^{\infty} a_n$ and

$s_k' = a_{n_0} + a_{n_0+1} + \dots + a_{n_0+k}$ be the partial sums of $\sum_{n=n_0}^{\infty} a_n$.

($\Longrightarrow$) Suppose $\sum_{n=1}^{\infty} a_k$ exists. Then,

$\lim_{k \to \infty} s_k = s$ for some $s \in \mathbb{C}$.

Note that

$$s_{n_0+k} = a_1 + a_2 + \dots + a_{n_0-1} + a_{n_0} + a_{n_0+1} + \dots + a_{n_0+k}$$
$$= a_1 + a_2 + \dots + a_{n_0-1} + s_k'$$
$$= w + s_k'$$

where $w = a_1 + a_2 + \dots + a_{n_0-1}$ is a fixed complex number.

So,

$$\lim_{k\to\infty} s_k' = \lim_{k\to\infty}\left(s_{n_0+k} - w\right) = \left(\lim_{k\to\infty} s_{n_0+k}\right) - w$$

$$= s - w.$$

Thus, $\sum_{n=n_0}^{\infty} a_n$ exists and

$$\sum_{n=n_0}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \overbrace{(a_1 + a_2 + \ldots + a_{n_0-1})}^{w}.$$

($\Leftarrow$) Now suppose $\sum_{n=n_0}^{\infty} a_n$ exists.

Then, $\lim_{k\to\infty} s_k' = s'$ for some $s' \in \mathbb{C}$.

As before we have $s_{n_0+k} = w + s_k'$.

Thus, $\lim_{k\to\infty} s_{n_0+k} = \lim_{k\to\infty}(w + s_k')$

$$= w + \lim_{k\to\infty} s_k' = w + s'.$$

Thus, $\lim_{k\to\infty} s_k = \lim_{k\to\infty} s_{n_0+k}$

$$= w + s'.$$

So, $\sum_{n=1}^{\infty} a_n$ converges and

$$\sum_{n=1}^{\infty} a_n = w + s' = a_1 + \dots + a_{n_0-1} + \sum_{n=n_0}^{\infty} a_n$$

③

Let

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

and

$$S_n' = \sum_{k=1}^{n} b_k = b_1 + b_2 + b_3 + b_4 + \cdots$$

be the partial sums for the two series.

Then $\lim_{n\to\infty} S_n = A$ and $\lim_{n\to\infty} S_n' = B$.

(a)

The partial sums for the series $\sum_{k=1}^{\infty}(a_k + b_k)$ are

$$S_n'' = \sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = S_n + S_n'.$$

Thus,

$$\lim_{n\to\infty} s_n'' = \lim_{n\to\infty} s_n + \lim_{n\to\infty} s_n' = A + B$$

property of convergent sequences

So, $\sum_{k=1}^{\infty} (a_k + b_k)$ converges to $A+B$.

(b) The partial sums of $\sum_{k=1}^{\infty} \alpha a_k$ are

$$s_n''' = \sum_{k=1}^{n} (\alpha a_k) = \alpha \sum_{k=1}^{n} a_k = \alpha s_n$$

Thus,

$$\lim_{n\to\infty} s_n''' = \lim_{n\to\infty} (\alpha s_n) = \alpha \lim_{n\to\infty} s_n$$

property of sequences

$= \alpha A.$ So, $\sum_{k=1}^{\infty} \alpha a_k$ converges to αA.

④

Let $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$ denote the n-th partial sum of the series.

($\Rightarrow$) Suppose that $\sum_{k=1}^{\infty} a_k$ converges.

Then $(s_n)_{n=1}^{\infty}$ converges.

Thus, $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Let $\varepsilon > 0$.

Then since (s_n) is a Cauchy sequence there exists $N > 0$ where if $n, m \geq N$ then $|s_m - s_n| < \varepsilon$. (*)

Let $n \geq N$ and $m = n + p$ where $p \geq 1$.

Then, $m \geq N$ also and

$$|s_m - s_n| = |s_{n+p} - s_n| = \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{n+p} a_k \right|$$

So (*) gives $\left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon$

($\Leftarrow$) Suppose that for every $\varepsilon > 0$ $\exists N > 0$ so that if $n \geq N$ then $\left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon$

for $p = 1, 2, 3, \ldots$

We can use this to show that $(S_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Let $\varepsilon > 0$.
Then from our assumption there exists $N > 0$ where if $n \geq N$ then $\left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon$

for $p = 1, 2, 3, \ldots$

Let $n, m \geq N$.
Without loss of generality suppose $m \geq n$.

Case 1: Suppose $m = n$.

Then,
$$|S_m - S_n| = |S_n - S_n| = 0 < \varepsilon$$

Case 2: Suppose $m > n$.

Then $m = n + p$ for some $p \geq 1$.

by assumption

So, $$|S_m - S_n| = \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon$$

From the two cases we see that given $m, n \geq N$, then $|S_m - S_n| < \varepsilon$.

So, $(S_n)_{n=1}^{\infty}$ is Cauchy.

Thus, $(S_n)_{n=1}^{\infty}$ converges.

Hence $\sum_{k=1}^{\infty} a_k$ converges. $\blacksquare$

⑤ Let s_n be the partial sums of $\sum_{k=1}^{\infty} a_k$ and s_n' be the partial sums of $\sum_{k=1}^{\infty} b_k$.

(a) Suppose $\sum b_k$ converges.

Let $\varepsilon > 0$.

By the Cauchy criterion for series (problem 4) there exists $N > 0$ so that if $n \geq N$ then

$$b_{n+1} + b_{n+2} + \cdots + b_{n+p} = |b_{n+1} + b_{n+2} + \cdots + b_{n+p}| < \varepsilon$$

for all $p \geq 1$.

$\boxed{b_k \text{ are positive real numbers}}$

Since $0 < a_k \leq b_k$ for all k we get that

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| = a_{n+1} + a_{n+2} + \cdots + a_{n+p} \leq b_{n+1} + b_{n+2} + \cdots + b_{n+p} < \varepsilon$$

for all $p \geq 1$. Thus, by the Cauchy criterion for series (problem 4), $\sum_{k=1}^{\infty} a_k$ converges

(b) (See here)

Easier: 5 (b) this is the converse of 5(a) and so is true. Here is a proof without seeing this

Suppose that $\sum_{k=1}^{\infty} a_k$ diverges.

Thus, $(s_n)_{n=1}^{\infty}$ diverges.

Note that since each $a_k > 0$, the sequence $s_n = a_1 + a_2 + \dots + a_n$ is an increasing sequence.

If $(s_n)_{n=1}^{\infty}$ was bounded, then by the monotone convergence theorem in real analysis (4650), the sequence s_n would have a limit.

Thus, $(s_n)_{n=1}^{\infty}$ is unbounded, ie $s_n \to \infty$ as $n \to \infty$.

Since $a_k \le b_k$ for all k, this tells us that

$$s_n = a_1 + a_2 + \dots + a_n \le b_1 + b_2 + \dots + b_n = s_n'$$

Thus, the sequence s_n' is also unbounded, ie $s_n' \to \infty$ as $n \to \infty$. So, $(s_n')_{n=1}^{\infty}$ does not converge and $\sum_{k=1}^{\infty} b_k$ diverges

⑥ (a) Consider the sequence $\sum_{n=1}^{\infty} \sin(\pi i^n)$

Note that

$$\sin(\pi i) = \frac{e^{i(\pi i)} - e^{-i(\pi i)}}{2i} = \frac{1}{2i}\left[e^{-\pi} - e^{\pi}\right]$$

$$= \frac{-i}{2}\left[e^{-\pi} - e^{\pi}\right] \neq 0$$

$$\sin(\pi i^2) = \sin(-\pi) = 0$$

$$\sin(\pi i^3) = \sin(-\pi i) = -\sin(\pi i) = \frac{i}{2}\left[e^{-\pi} - e^{\pi}\right] \neq 0$$

4680 HW
$\sin(-z) = -\sin(z)$

$$\sin(\pi i^4) = \sin(\pi) = 0$$

$$\sin(\pi i^5) = \sin(\pi i) = \frac{-i}{2}\left[e^{-\pi} - e^{\pi}\right]$$

$i^4 = 1$

$$\sin(\pi i^6) = \sin(\pi i^2) = 0$$

$\vdots \quad \vdots$

The terms alternate between the above four numbers and hence don't go to 0. By the divergence thm, this series diverges.

6 (b)

Note that

$$\left|\frac{1+(-i)^n}{n^2}\right| = \frac{|1+(-i)^n|}{|n^2|} \overset{\triangle\text{-inequality}}{\leq} \frac{|1|+|(-i)^n|}{n^2}$$

$$\underset{|(-i)^n| = |-i|^n = 1^n = 1}{=} \frac{2}{n^2}$$

And, $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Thus, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{1+(-i)^n}{n^2}\right|$ converges by the $\underbrace{\text{comparison test}}_{\text{(Problem 5)}}$ since $\left|\frac{1+(-i)^n}{n^2}\right| \leq \frac{2}{n^2}$ for all n.

So, $\displaystyle\sum_{n=1}^{\infty}\frac{1+(-i)^n}{n^2}$ converges absolutely.

(6) (c) Suppose $|z|<1$.

Then, $\sum_{n=1}^{\infty} |z^n| = \sum_{n=1}^{\infty} |z|^n$

$= |z| + |z|^2 + |z|^3 + \dots$

$= |z| \left[1 + |z| + |z|^2 + \dots \right] = |z| \frac{1}{1-|z|}$

from class

Thus, $\sum_{n=1}^{\infty} z^n$ converges absolutely if $|z|<1$.

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(6) (d) Suppose $|z| \geq 1$.

If $|z|=1$, then

$\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = \lim_{n\to\infty} 1^n = 1$

If $|z|>1$, then

$$\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = \infty$$

In either case

$$\lim_{n\to\infty} |z^n| \neq 0$$

Thus, if $|z|\geq 1$, then

$$\lim_{n\to\infty} z^n \neq 0$$

Recall

$$\lim_{n\to\infty} a_n = 0$$

iff

$$\lim_{n\to\infty} |a_n| = 0$$

So, by the divergence test

$$\sum_{n=1}^{\infty} z^n$$

diverges if $|z|\geq 1$.

⑦

Let $s_n = \sum_{k=1}^{n} a_k$ be the n-th partial sum of the series.

Since $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{n\to\infty} s_n = s$ for some $s \in \mathbb{C}$.

Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[(a_1+a_2+\dots+a_n)-(a_1+a_2+\dots+a_{n-1})\right]$$

$$= \lim_{n\to\infty}\left[s_n - s_{n-1}\right]$$

$$= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}$$

$$= s - s = 0$$

8 (a)

Consider the k-th partial sum

$$S_k = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

We will show that $(S_k)_{k=1}^{\infty}$ is unbounded and hence cannot converge.

We look at a sub-series.

Note that

$$S_{2^0} = S_1 = 1$$

$$S_{2^1} = S_2 = 1 + \frac{1}{2}$$

$$S_{2^2} = S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$\geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{2}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$$

$$S_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$\geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{2} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{4}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3 \cdot \frac{1}{2}$$

In general,

$$S_{2^k} \geq 1 + k \cdot \frac{1}{2}$$

Thus, $S_{2^k} \to \infty$ as $k \to \infty$.

Therefore, $(S_k)_{k=1}^{\infty}$ is unbounded and thus diverges.

So, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

8 (b) Let $p \in \mathbb{R}$ with $p \leq 1$.

<u>case 1:</u> Suppose $p = 1$. Then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by 8(a).

<u>case 2:</u> Suppose $p < 1$.

Then, $\frac{1}{n^p} > \frac{1}{n}$ for all $n \geq 1$.

Thus, by the comparison test since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{1}{n^p}$.