5680
HW 3
Solutions

① (a) Consider $\sum_{n=1}^{\infty} n^2 z^n$

We have

$$r = \lim_{n\to\infty} \left| \frac{(n+1)^2 z^{n+1}}{n^2 z^n} \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| |z|$$

$$= \lim_{n\to\infty} |z| \cdot \left[\frac{n^2+2n+1}{n^2} \right] = |z| \cdot 1 = |z|$$

Thus, by the ratio test, if $|z|<1$ then $\sum_{n=1}^{\infty} n^2 z^n$ converges and if $|z|>1$ the series diverges.

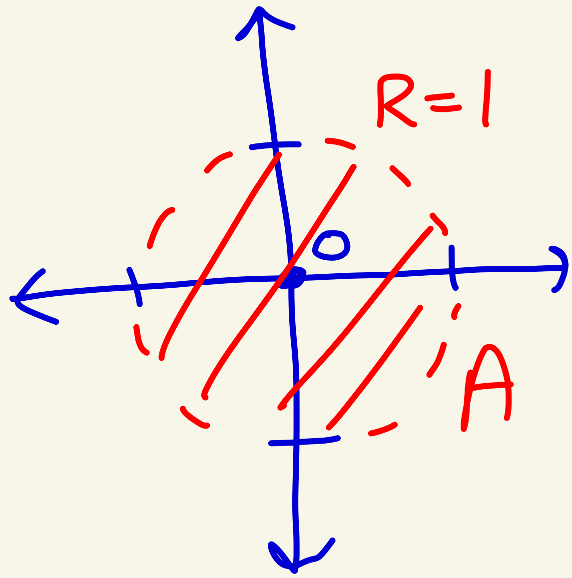
So the radius of convergence is $R=1$.

Note that if $|z|=1$, then $\lim_{n\to\infty} |n^2 z^n| = \lim_{n\to\infty} n^2 = \infty$

So, the series diverges if $|z|=1$.

The set that the series converges on is therefore

$A = \{ z \mid |z| < 1 \}$



I did a little more than the question asked just to show you some more info.

① (b) Consider $\sum_{n=1}^{\infty} n! \frac{z^n}{n^n}$

Note that

$$\lim_{n\to\infty} \left| \frac{(n+1)! \frac{z^{n+1}}{(n+1)^{n+1}}}{n! \frac{z^n}{n^n}} \right| = \lim_{n\to\infty} |z| \cdot \left| (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} \right|$$

$$\boxed{(n+1)! = (n+1) \cdot [n!]}$$

$$= \lim_{n\to\infty} |z| \cdot \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} |z| \cdot \left(\frac{n}{n+1}\right)^n \quad (*)$$

Let's calculate $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$.

Note that

$$\left(\frac{n}{n+1}\right)^n = e^{\ln\left[\left(\frac{n}{n+1}\right)^n\right]} = e^{n \cdot \ln\left(\frac{n}{n+1}\right)}$$

Note that

$$\lim_{n\to\infty} n\cdot\ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}$$

$$\boxed{\begin{array}{l}\ln\left(\frac{n}{n+1}\right)\to\ln(1)=0\\ \frac{1}{n}\to 0\end{array}}$$

$$\overset{L'H}{\underset{\text{"}\frac{0}{0}\text{"}}{=}} \lim_{n\to\infty} \frac{\frac{1}{\left(\frac{n}{n+1}\right)}\cdot\frac{1\cdot(n+1)-n\cdot(1)}{(n+1)^2}}{-\frac{1}{n^2}}$$

$\boxed{\text{L'Hospital rule}}$

$$= \lim_{n\to\infty}\left[-n^2\cdot\left(\frac{n+1}{n}\right)\cdot\frac{1}{(n+1)^2}\right] = \lim_{n\to\infty}\left[\frac{-n^2}{n^2+n}\right]$$

$$= \lim_{n\to\infty}\frac{-1}{1+\frac{1}{n}} = \frac{-1}{1+0} = -1.$$

Thus,

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} e^{n\cdot\ln\left(\frac{n}{n+1}\right)}$$

$$= e^{\lim_{n\to\infty} n\cdot\ln\left(\frac{n}{n+1}\right)} = e^{-1} = \frac{1}{e}$$

Thus,

$$\lim_{n\to\infty} |z|\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{|z|}{e}$$

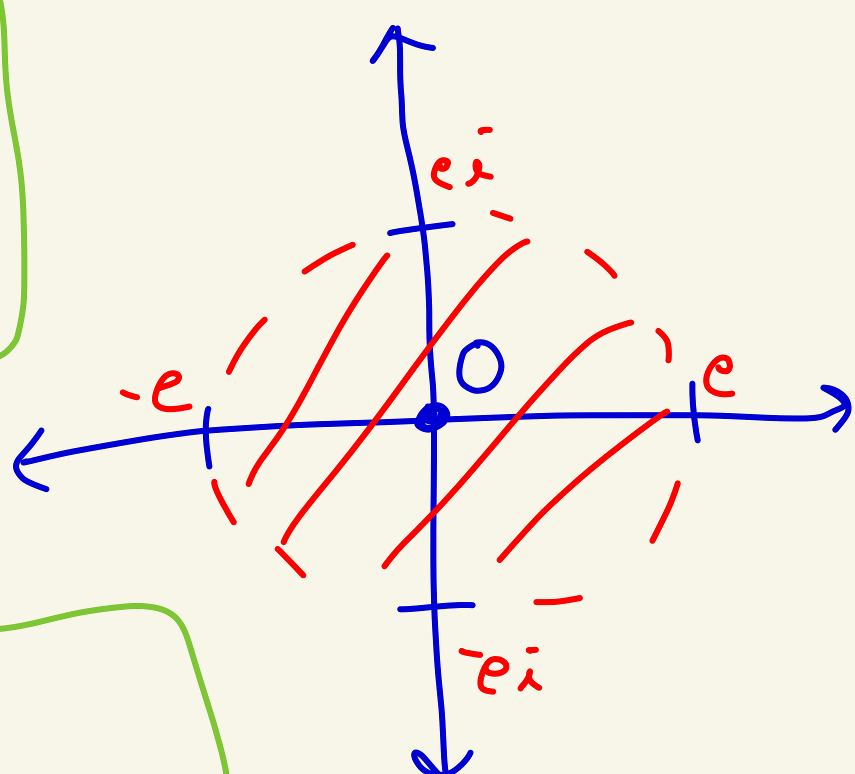
And $\frac{|z|}{e} < 1$ iff $|z| < e$.

Thus, the radius of convergence is $R = e$.

The power series converges for $|z| < e$ and diverges for $|z| > e$.

When $|z| = e$ it's unknown.

You could try plugging in some boundary points to see what happens.



We at least get convergence here. You can try plugging in boundary pts to see what happens

① (c) Consider $\sum_{n=0}^{\infty} \frac{z^{2n}}{4^n}$

We have

$$\lim_{n\to\infty} \left| \frac{z^{2(n+1)}}{4^{n+1}} \cdot \frac{4^n}{z^{2n}} \right| = \lim_{n\to\infty} \left| \frac{z^2}{4} \right| = \frac{|z|^2}{4}$$

Note that $\frac{|z|^2}{4} < 1$ iff $|z|^2 < 4$

iff $|z| < 2$

Thus, the radius of convergence is $R = 2$.

① (d)

Consider $\displaystyle\sum_{n=0}^{\infty} \frac{(z-1)^n}{1+2^n}$

Note that

$$\lim_{n\to\infty} \left| \frac{(z-1)^{n+1}}{1+2^{n+1}} \cdot \frac{1+2^n}{(z-1)^n} \right| = \lim_{n\to\infty} |z-1| \cdot \frac{1+2^n}{1+2^{n+1}}$$

$$\lim_{n\to\infty} |z-1| \left[\frac{\frac{1}{2^{n+1}} + \frac{1}{2}}{\frac{1}{2^{n+1}} + 1} \right] = |z-1| \cdot \left[\frac{0+\frac{1}{2}}{0+1} \right]$$

$$= \frac{|z-1|}{2}$$

And $\frac{|z-1|}{2} < 1$ iff $|z-1| < 2$.

Thus, $R = 2$ is the radius of convergence.

(2)(a) $f(z) = e^z$, $z_0 = 1$

$f'(z) = e^z$, $f'(1) = e$

$f''(z) = e^z$, $f''(1) = e$

$\vdots \qquad \vdots$

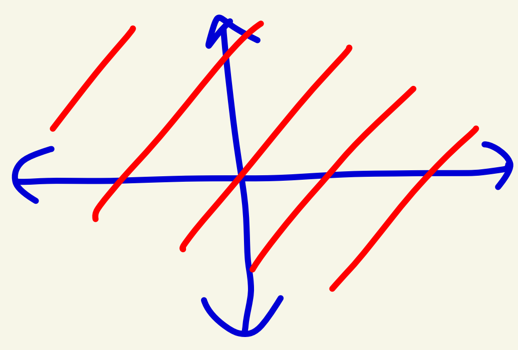
$f^{(n)}(z) = e^z$, $f^{(n)}(1) = e$

Thus, the taylor series for e^z centered at $z_0 = 1$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n$$

Since $f(z) = e^z$ is analytic on $A = \mathbb{C}$, by Taylor's thm, the above series converges for all $z \in \mathbb{C}$. So, its radius of convergence is $R = \infty$

Converges on all of $\mathbb{C}$ →



(2) (b) $f(z) = \frac{1}{z}$, $z_0 = 1$

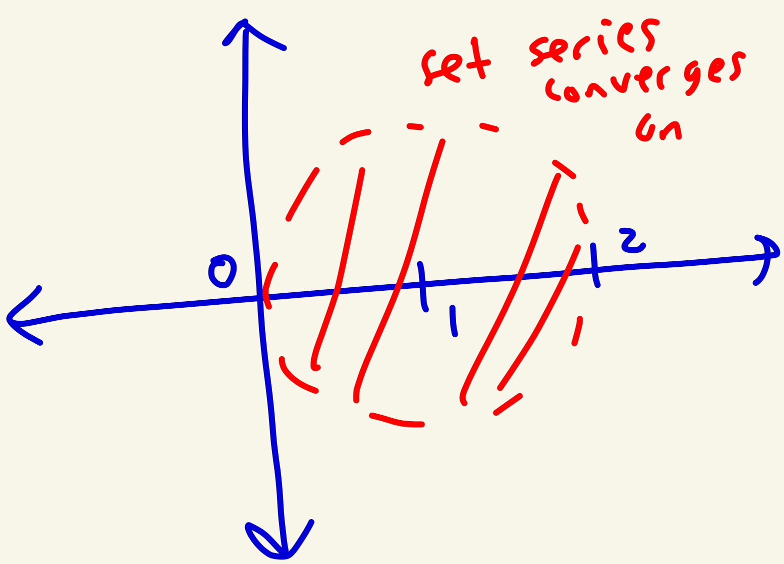
$\boxed{\text{Method 1°}}$

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \frac{1}{1-(-(z-1))}$$

$$= \sum_{n=0}^{\infty} \left[-(z-1)\right]^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

equals iff $|-(z-1)|<1$
iff $|z-1|<1$

This series is a geometric series which converges iff $|-(z-1)|<1$ iff $|z-1|<1$



$\boxed{\text{Method 2:}}$

$f(z) = z^{-1}$, $f(1) = 1$

$f'(z) = -z^{-2}$, $f'(1) = -1$

$f''(z) = 2z^{-3}$, $f''(1) = 2!$

$f^{(3)}(z) = -3 \cdot 2 \cdot z^{-4}$, $f^{(3)}(1) = -3!$

$\vdots$ $\vdots$

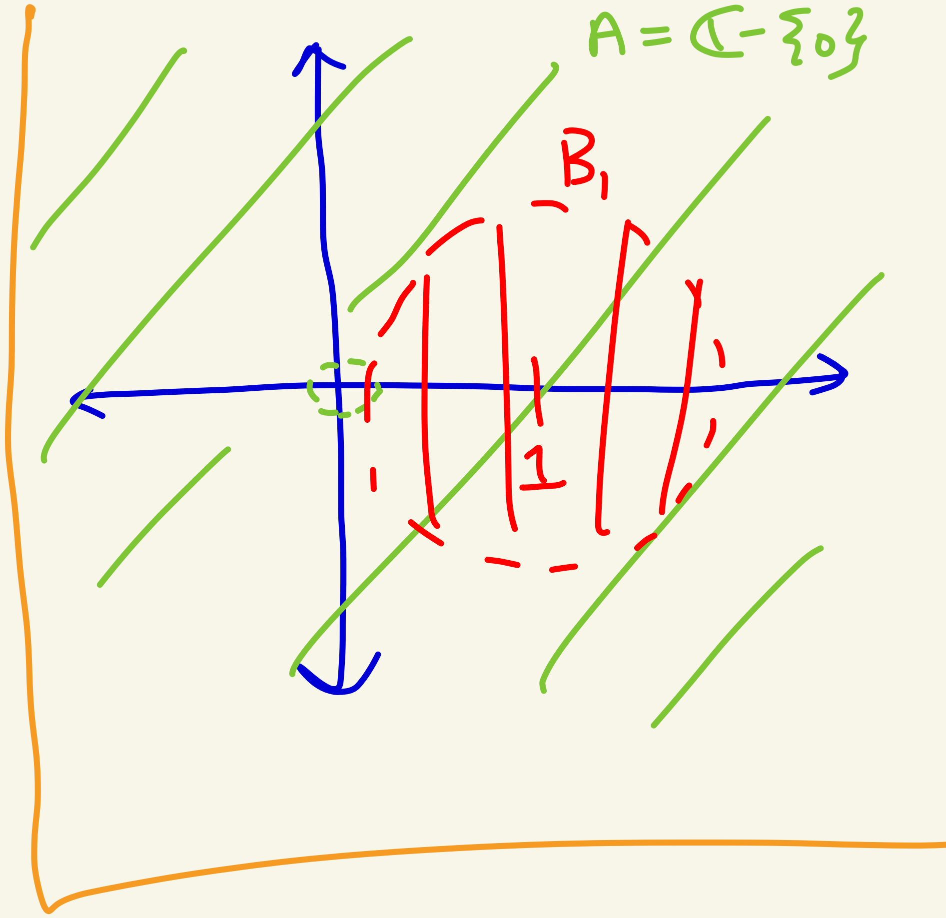
$f^{(n)}(z) = \dfrac{(-1)^n \cdot n!}{z^{n+1}}$, $f^{(n)}(1) = (-1)^n \cdot n!$

So, the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n!}{n!} \cdot (z-1)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Note that $f(z) = \frac{1}{z}$ is analytic on $A = \mathbb{C} - \{0\}$.



And,

$B_1 = \{ z \mid |z-1| < 1 \}$

is contained in A and centered at 1.

So, by Taylor's Thm

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

when $z \in B_1$

This is a geometric series $\left[\sum_{n=0}^{\infty} (-(z-1))^n \right]$

so it converges

iff $|-(z-1)| < 1$ iff $|z-1| < 1$.

So, B_1 is the set that the series converges on.

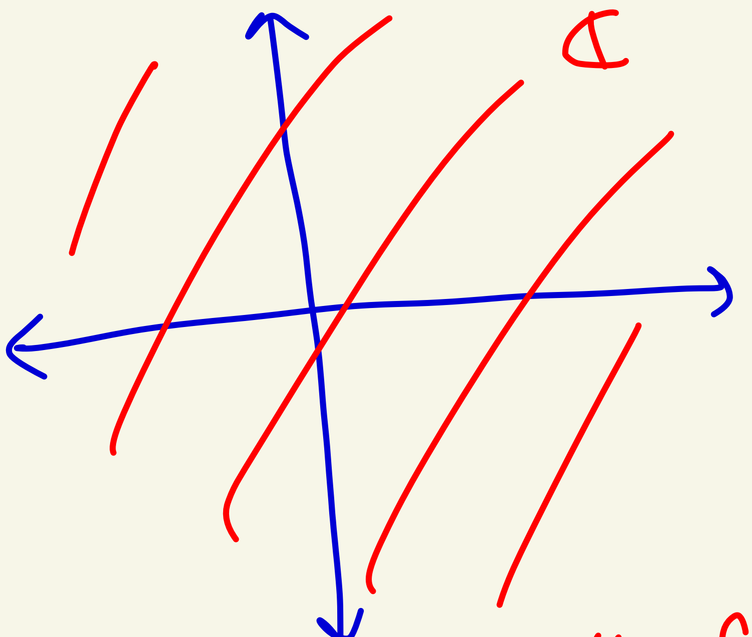
2 (c)

$$f(z) = e^{z^2}, \quad z_0 = 0$$

We know that $e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z \in \mathbb{C}$.

Thus, $$e^{z^2} = \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$$

which converges for all $z \in \mathbb{C}$.



converges on all of $\mathbb{C}$

(2) (d)

We know that
$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{for all } z \in \mathbb{C}$$

Thus,
$$\sin(z^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(z^2)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

(2) (e) Note that $f(z) = z^2 + z$ is analytic at $z_0 = 1$. So we can make a power series centered at $z_0 = 1$.

We will use the formula for the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n$

We have

$$f(z) = z^2 + z \qquad f(1) = 2$$
$$f'(z) = 2z + 1 \qquad f'(1) = 3$$
$$f''(z) = 2 \qquad f''(1) = 2$$
$$f'''(z) = 0 \qquad f'''(1) = 0$$

And $f^{(n)}(1) = 0$ for $n \geq 3$.

So,

$$z^2 + z = f(1) + \frac{f'(1)}{1}(z-1) + \frac{f''(1)}{2!}(z-1)^2$$
$$= 2 + 3(z-1) + (z-1)^2$$

This power series converges for all $z \in \mathbb{C}$.

(3) $f(z) = \dfrac{1}{(z-1)(z-2)}$, $z_0 = 0$

$\boxed{\text{Method 1:}}$

Use partial fractions:

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Must be true for all z

If $z=2$, then $1 = A(0) + B(1)$

So, $B = 1$.

If $z=1$, then $1 = A(-1) + B(0)$

So, $A = -1$.

plug in specific z to find A and B

So,

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} + \frac{-1/2}{1-\frac{z}{2}}$$

Note that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ iff $|z|<1$

And $\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{2^n}$ iff $\left|\frac{z}{2}\right|<1$

iff $|z|<2$

Thus, when $|z|<1$ <u>and</u> $|z|<2$, ie when $|z|<1$ we get that

$$\frac{1}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{1}{2}\cdot\left[\frac{1}{1-\frac{z}{2}}\right]$$

$$= \left(1+z+z^2+z^3+\dots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+\dots\right)$$

$$= \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n$$

Method 2:

$$\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1}\right)\left(\frac{1}{z-2}\right) = \left(\frac{-1}{1-z}\right)\left(\frac{-1/2}{1-z/2}\right)$$

Note that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ iff $|z|<1$

And $\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{2^n}$ iff $\left|\frac{z}{2}\right|<1$ iff $|z|<2$

Thus, when $|z|<1$ <u>and</u> $|z|<2$, ie when $|z|<1$ we get that

$$\frac{1}{(z-1)(z-2)} = \frac{1}{2}\left[\sum_{n=0}^{\infty} z^n\right]\left[\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1^k \cdot \frac{1}{2^{n-k}}\right) z^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^{n-k}}\right) z^n$$

If you don't like this formal way, you can write out several terms and FOIL

$\left(\sum_{n=0}^{\infty} a_n z^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} c_n z^n$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

(Thm from class)

Note that

$$\sum_{k=0}^{n} \frac{1}{2^{n-k}} = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \dots + \left(\frac{1}{2}\right) + 1$$

$$= \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2 \cdot \left[1 - \frac{1}{2^{n+1}}\right]$$

$$\boxed{1 + z + z^2 + \dots + z^n = \frac{1-z^{n+1}}{1-z}}$$

So when $|z|<1$ we have

$$\frac{1}{(z-1)(z-2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{2^{n-k}} \right) z^n$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(2 \cdot \left[1 - \frac{1}{2^{n+1}}\right] \right) z^n$$

$$= \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n$$

Note we got the same answer as in method 1

④ (a)

$$f(z) = z^{-1} \cdot \sin(z), \quad f(1) = \sin(1)$$

$$f'(z) = -z^{-2} \cdot \sin(z) + z^{-1} \cdot \cos(z),$$

$$f(1) = -\sin(1) + \cos(1)$$

$$f''(z) = 2z^{-3} \cdot \sin(z) - z^{-2}\cos(z)$$
$$- z^{-2}\cos(z) - z^{-1}\sin(z)$$

$$f''(1) = 2\sin(1) - \cos(1) - \cos(1) - \sin(1)$$
$$= -2\cos(1) + \sin(1)$$

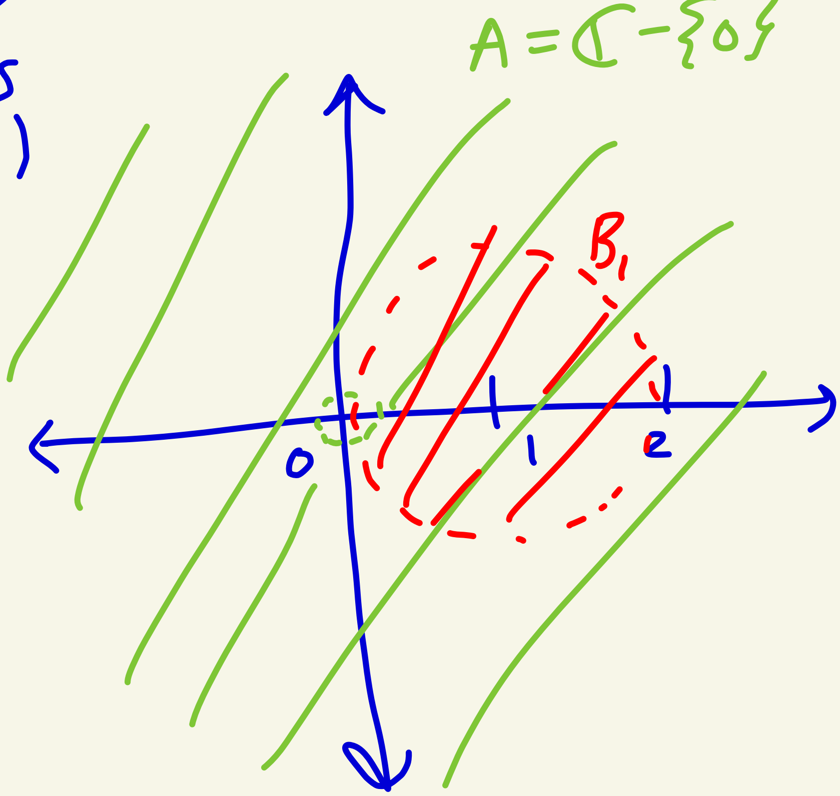
So, the first few terms of the taylor series are

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = \sin(1)$$
$$+ \left[-\sin(1) + \cos(1)\right](z-1)$$
$$- \frac{2\cos(1) + \sin(1)}{2!}(z-1)^2$$
$$+ \cdots$$

Note that $f(z) = \frac{\sin(z)}{z}$ is analytic on $A = \mathbb{C} - \{0\}$.

So, by Taylor's Thm, this power series converges to $f(z)$ on

$B_1 = \{ z \mid |z-1| < 1 \}$



(4) (b) $f(z) = e^z \sin(z)$, $z_0 = 0$

We have that $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$ for all z.

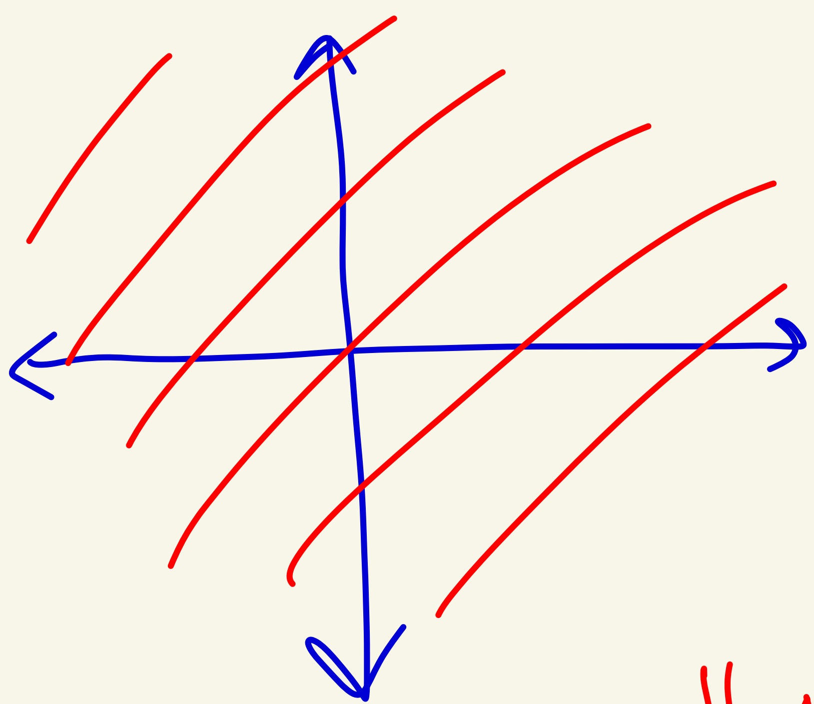
And $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ for all z.

Thus, for all z we have

$$e^z \sin(z) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\right)$$

$$= z + z^2 + \left(-\frac{1}{6} + \frac{1}{2}\right) z^3 + \left(-\frac{1}{6} + \frac{1}{6}\right) z^4 + \cdots$$

$$= z + z^2 + \frac{1}{3} z^3 + 0 z^4 + \cdots.$$



Converges on all of $\mathbb{C}$

⑤ (a) Let $f(z) = \frac{1}{(1-z)^2}$

We know that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z|<1$.

That is, it has radius of convergence $R=1$.

Differentiating both sides we get

$$\frac{1}{(1-z)^2} = \left[\frac{1}{1-z}\right]' = \sum_{n=0}^{\infty} n z^{n-1} \text{ for } |z|<1$$

That is,

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}$$

with radius of convergence $R=1$

5 (b) This is the same idea as 5(a).

We have

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} \quad \text{for } |z|<1$$

ie radius of convergence $R=1$.

By differentiating both sides we have

$$\frac{2}{(1-z)^3} = \sum_{n=1}^{\infty} n(n-1) z^{n-2} \quad \text{for } |z|<1$$

ie radius of convergence $R=1$

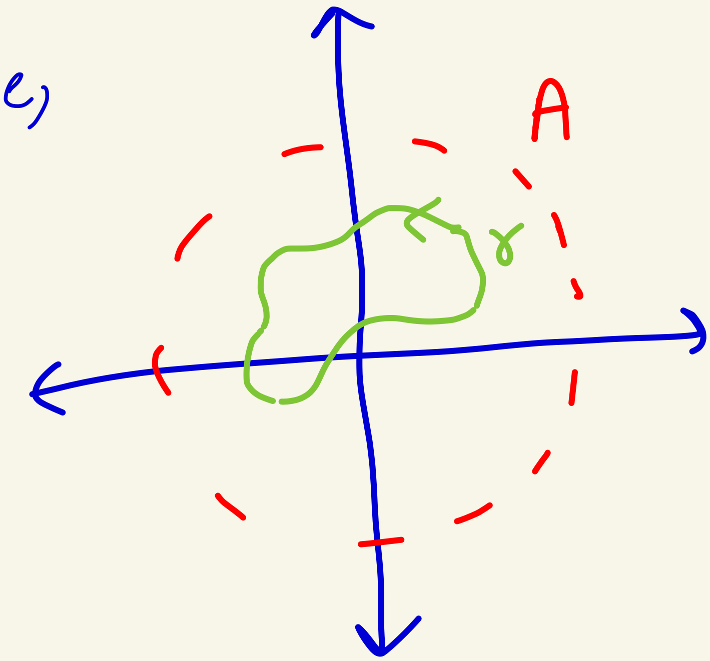
So,

$$\frac{1}{(1-z)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1) z^{n-2} \quad \text{for } |z|<1$$

ie radius of convergence $R=1$

⑥ Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R > 0$. Let $A = \{ z \mid |z| < R \}$.

Then f is analytic in A.



Since γ is a simple, piecewise smooth, closed curve lying inside of A,

by Cauchy's Theorem (Math 4680),

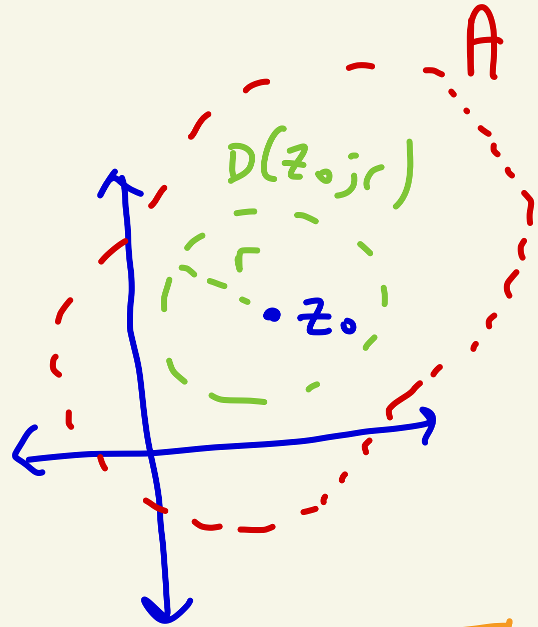
$$\int_{\gamma} f = 0.$$

7 Let $A \subseteq \mathbb{C}$ be open.

Let f be analytic on A.

Let $z_0 \in A$ with $f(z_0)=0$.

Since A is open, there exists $r>0$ where $D(z_0;r) \subseteq A$.



Then, by Taylor's thm,

$$\boxed{f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \text{ for } z \in D(z_0;r).} \quad (*)$$

<u>case (i)</u>: If $f^{(k)}(z_0)=0$ for all $k \geq 0$, then by (*) we know $f(z)=0$ for all $z \in D(z_0;r)$.

<u>case (ii)</u>: Otherwise, there exists a smallest integer $n \geq 0$ where $f^{(n)}(z_0) \neq 0$

If $n=0$, then $f(z_0) = f^{(0)}(z_0) \neq 0$.

But we assumed $f(z_0)=0$.

Thus $n>0$ must be true.

Thus, $n>0$ and

$$f(z) = \underbrace{\frac{f^{(n)}(z_0)}{n!}}_{\text{not } 0}(z-z_0)^n + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0)^{n+1} + \cdots$$

$$= (z-z_0)^n \left[\underbrace{\frac{f^{(n)}(z_0)}{n!}}_{\text{not } 0} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0) + \cdots \right]$$

$$= (z-z_0)^n \varphi(z)$$

where $\varphi(z) = \frac{f^{(n)}(z_0)}{n!} + \frac{f^{(n+1)}(z_0)}{(n+1)!}(z-z_0) + \cdots$

and $\varphi(z_0) = \frac{f^{(n)}(z_0)}{n!} \neq 0$.

We showed in class that $\varphi(z)$ is analytic at z_0.

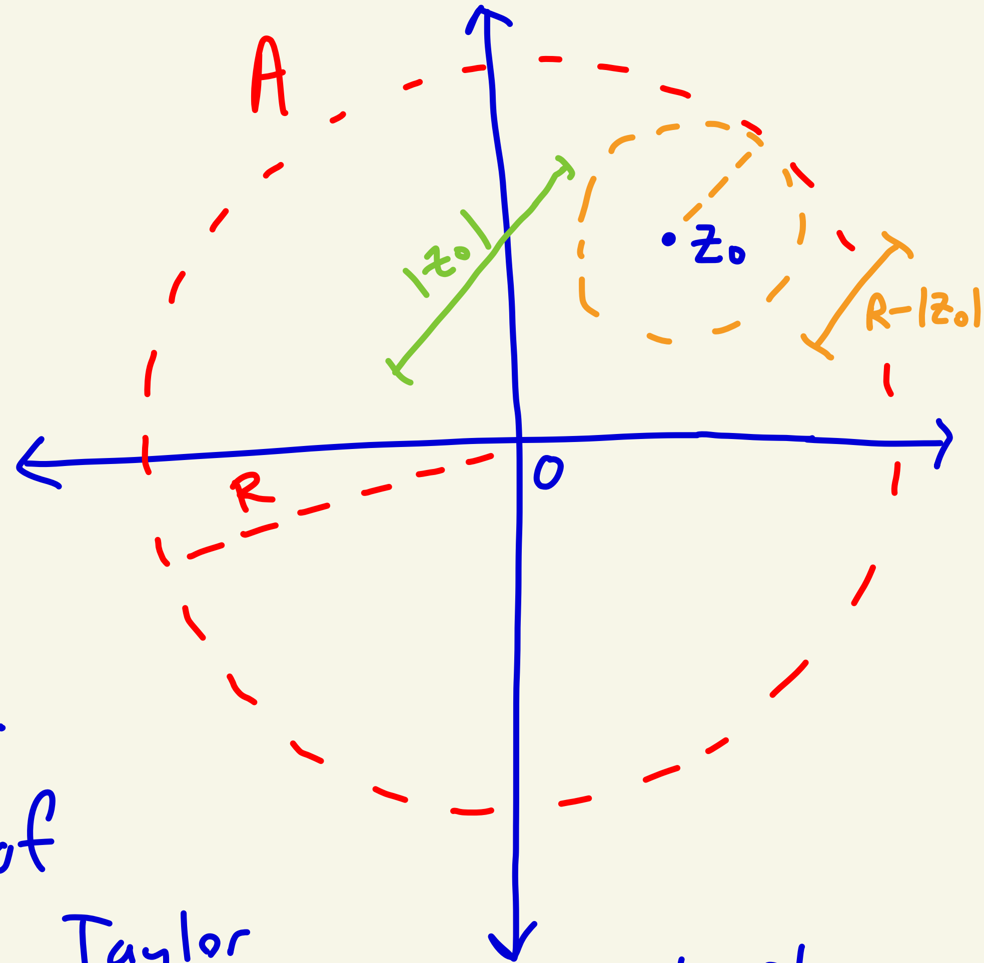
Since φ is then continuous at z_0 and $\varphi(z_0) \neq 0$, by Math 4680 HW 4 #5, there exists $\hat{r} > 0$ where $\varphi(z) \neq 0$ for all $z \in D(z_0; \hat{r})$, here we take $\hat{r} < r$.

Thus, if $z \in D(z_0; \hat{r}) - \{z_0\}$, then

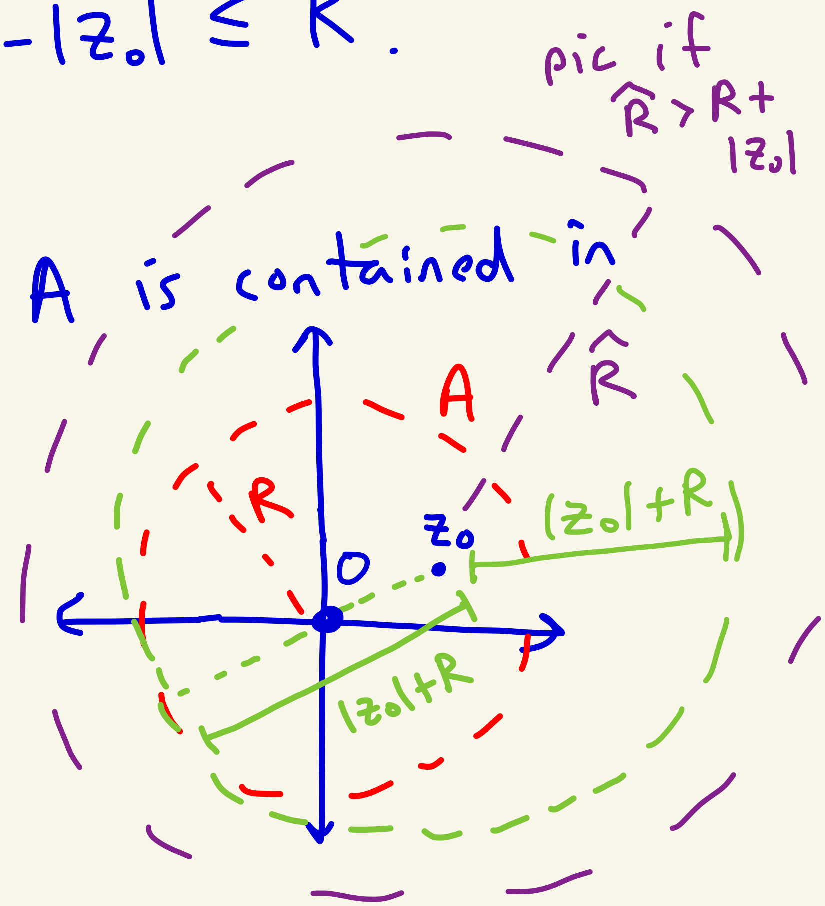
$$f(z) = (z - z_0)^n \varphi(z) \neq 0.$$

8

Note that the disk centered at z_0 of radius $R-|z_0|$ is contained in A.



Thus, the radius of convergence of the Taylor series centered at z_0 must be at least $R-|z_0|$. Thus, $R-|z_0| \leq \hat{R}$.



On the other hand, the disc of radius $R+|z_0|$ centered at z_0.

If $\hat{R} > R+|z_0|$ then f would be analytic on a strictly bigger disc than A centered at z_0. But this isn't true since R is the radius of convergence of f centered at 0, which is the largest disc centered at 0 that f is analytic on.

Thus, $\hat{R} \le R + |z_0|$.

So, $R - |z_0| \le \hat{R} \le R + |z_0|$