5680

HW 4

Part 2

Solutions

1 (a)

Note that both $e^z - 1$ and $\sin(z)$ are analytic at $z_0 = 0$.
Their power series expansions there are

$$e^z - 1 = -1 + e^z = -1 + \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$

$$= z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)$$

$$= z\,\varphi_1(z)$$

where $\varphi_1(0) = 1 + \frac{0}{2!} + \frac{0^2}{3!} + \cdots = 1 \neq 0.$

Thus, $e^z - 1$ has a zero of order 1 at $z_0 = 0$.

And about $z_0 = 0$ we have

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)$$

$$= z\,\varphi_2(z)$$

where $\varphi_2(0) = 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \cdots = 1 \neq 0$

Thus, $\sin(z)$ has a zero of order 1 at $z_0 = 0$.

Since $e^z - 1$ and $\sin(z)$ are both analytic and they both have zeros of order 1 at $z_0 = 0$, by a theorem from class

$$f(z) = \frac{e^z - 1}{\sin(z)}$$

has a removable singularity at $z_0 = 0$.

Thus, $\text{Res}(f; 0) = 0$.

If you wanted to you could also write

$$\frac{e^z - 1}{\sin(z)} = \frac{\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)} = \frac{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}$$

and then dividing denominator into numerator to find the power series expansion at $z_0 = 0$.

1 (b)

$\boxed{\text{Method 1}}$

$$f(z) = \frac{1}{e^z - 1} = \frac{g(z)}{h(z)}$$

where $g(z) = 1$ and $h(z) = e^z - 1$.

The numerator satisfies $g(0) = 1 \neq 0$.

The denominator satisfies $h(0) = e^0 - 1 = 1 - 1 = 0$.

Also, $h(z) = e^z - 1$ is analytic at $z_0 = 0$ with power series expansion

$$h(z) = e^z - 1 = -1 + \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$$

$$= z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= z\underbrace{\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}_{\text{let this be } \varphi(z)} = z\,\varphi(z)$$

where $\varphi(z)$ is analytic at $z_0 = 0$ and

$$\varphi(0) = 1 + \frac{0}{2!} + \frac{0^2}{3!} + \cdots = 1 \neq 0$$

So, $f(z) = \dfrac{1}{e^z - 1}$ where the numerator has no zero at $z_0 = 0$ and the denominator has a zero at $z_0 = 0$ of order 1.

By a theorem from class

f has a pole of order 1 at $z_0 = 0$

And

$$\operatorname{Res}(f;0) = \lim_{z \to 0} (z-0) f(z)$$

$$= \lim_{z \to 0} \frac{z}{e^z - 1}$$

$$= \lim_{z \to 0} \frac{z}{z\,\varphi(z)}$$

$$= \lim_{z \to 0} \frac{1}{\varphi(z)}$$

$$= \frac{1}{\left(1 + \frac{0}{2!} + \frac{0^2}{3!} + \cdots\right)} = \frac{1}{1} = 1.$$

① (b)

$\boxed{\text{Method 2}}$

Here we have $f(z) = \dfrac{1}{e^z - 1}$

and the numerator is not 0 at $z_0 = 0$
but the denominator is, ie $e^0 - 1 = 1 - 1 = 0$.
So we will have either a pole of order m
at $z_0 = 0$ or an essential singularity there.
let's divide the numerator by the
denominator to get the Laurent series
at $z_0 = 0$.

We know $e^z - 1 = -1 + \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)$

$$= z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots$$

Thus,

$$\frac{1}{z} - \frac{1}{2} + \frac{1}{12} z + \cdots$$

$$\left(z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots\right) \overline{\Big) \; 1 \qquad\qquad\qquad\qquad}$$

$$-\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \cdots\right)$$

$$\overline{-\frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} - \cdots}$$

$$-\left(-\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{12} - \cdots\right)$$

$$\overline{\frac{1}{12} z^2 + \frac{1}{24} z^3 + \cdots}$$

$$-\left(\frac{1}{12} z^2 + \cdots\right)$$

$$\overline{\vdots \qquad \vdots}$$

residue

So, in a deleted neighborhood of $z_0 = 0$ we have

$$f(z) = \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12} z + \cdots$$

So we have a pole of order 1 with $\operatorname{Res}(f; 0) = 1$.

① (c) $f(z) = \dfrac{z+2}{z^2-2z}$ at $z_0 = 0$

Note that the numerator $z+2$ has no zero at $z_0 = 0$ since $0+2 = 2 \neq 0$.

The denominator has a zero at $z_0 = 0$ since $0^2 - 2 \cdot 0 = 0$. It has a zero of order 1 since

$$z^2 - 2z = \underset{\substack{\uparrow \\ \text{zero of order 1 at 0.}}}{z} \underbrace{(z-2)}_{\text{not 0 at } z_0 = 0}.$$

So we will have a simple pole at $z_0 = 0$, ie a pole of order 1.

Another way to see this is to notice that

$$f(z) = \frac{z+2}{z^2-2z} = \frac{\left(\frac{z+2}{z-2}\right)}{z} = \frac{\varphi(z)}{z}$$

where $\varphi(z) = \dfrac{z+2}{z-2}$ is analytic at $z_0 = 0$ and $\varphi(0) = \dfrac{0+2}{0-2} = -1 \neq 0$

So, we have a pole of order 1 at $z_0 = 0$.

Furthermore, from a theorem in class

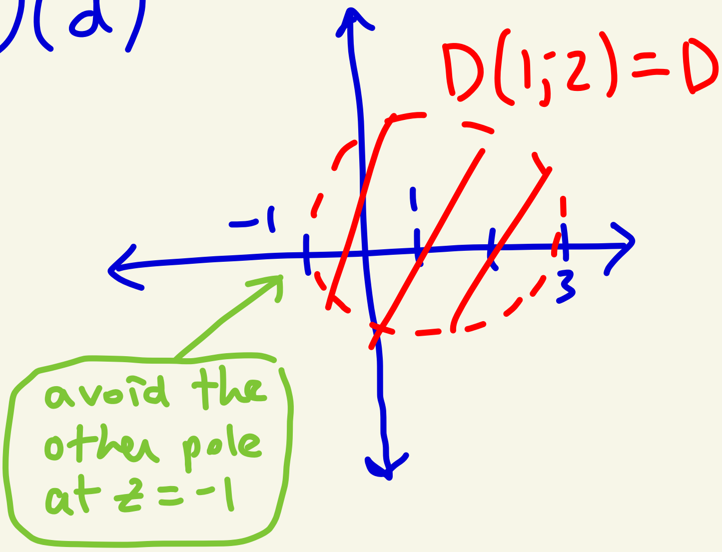
$$\operatorname{Res}(f; 0) = \varphi(0) = -1.$$

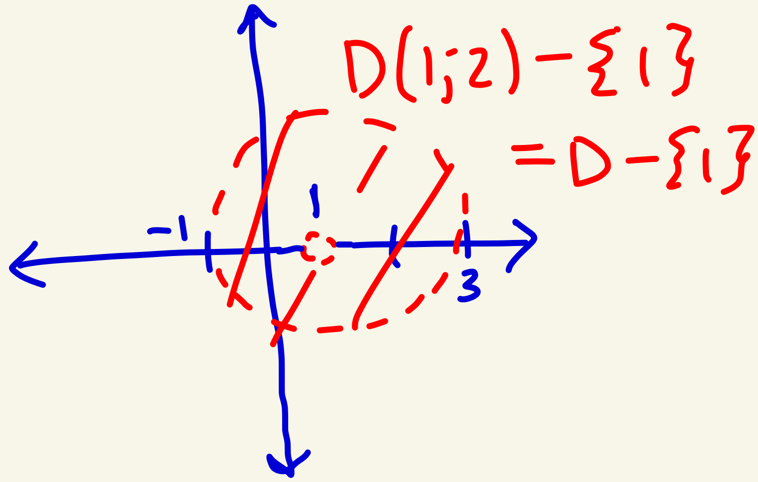
Class thm: Suppose f has a pole of order m at z_0 and $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ is some deleted neighborhood of z_0 and φ is analytic at z_0 and $\varphi(z_0) \neq 0$.

If $m = 1$, then $\operatorname{Res}(f; z_0) = \varphi(z_0)$

If $m \geq 2$, then $\operatorname{Res}(f; z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

1(d)





Let $D = D(1;2)$.

Let $z \in D(1;2) - \{1\}$.

Then,

$$f(z) = \frac{e^z}{(z^2-1)^2} = \frac{e^z}{[(z+1)(z-1)]^2}$$

$$= \frac{\left(\frac{e^z}{(z+1)^2}\right)}{(z-1)^2} = \frac{\varphi(z)}{(z-1)^2}$$

where $\varphi(z)$ is analytic at $z_0 = 1$, indeed in all of D. And $\varphi(1) \neq 0$.

By the thm in class [which is also written down in the solutions for problem 1(c)] we have that

f has a pole of order 2 at $z_0 = 1$ and

$$\text{Res}(f;1) = \frac{\varphi^{(2-1)}(1)}{(2-1)!}$$

$$= \frac{\varphi^{(1)}(1)}{1!} = \varphi'(1)$$

Note that $\varphi'(z) = \dfrac{e^z(z+1)^2 - 2(z+1)e^z}{(z+1)^4}$

So, $\text{Res}(f;1) = \varphi'(1) = \dfrac{e^1(1+1)^2 - 2(1+1)e^1}{(1+1)^4}$

$$= \frac{4e - 4e}{2} = 0$$

① (e)

$$f(z) = \frac{e^{z^2}}{(z-1)^4} = \frac{\varphi(z)}{(z-1)^4}$$

where φ is analytic at $z_0 = 1$ and $\varphi(1) = e^{1^2} \neq 0$.

By a thm in class, we have a pole of order $m = 4$ and

$$\mathrm{Res}(f;1) = \frac{\varphi^{(4-1)}(1)}{(4-1)!} = \frac{\varphi^{(3)}(1)}{3!}$$

We have $\varphi(z) = e^{z^2}$

$$\varphi'(z) = 2ze^{z^2}$$

$$\varphi''(z) = 2e^{z^2} + 2z(e^{z^2} \cdot 2z)$$

$$= 2e^{z^2} + 4z^2e^{z^2}$$

$$\varphi'''(z) = 2 \cdot 2ze^{z^2} + 8ze^{z^2} + 4z^2(e^{z^2} \cdot 2z)$$

$$= 4ze^{z^2} + 8ze^{z^2} + 8z^3e^{z^2}$$

So,

$$\mathrm{Res}(f;1) = \frac{\varphi'''(1)}{6} = \frac{4(1)e^1 + 8(1)e^1 + 8(1)e^1}{6} = \frac{20e}{6}$$

$$= \frac{10}{3}e$$

① (f)

In this case

$$f(z) = \frac{z^2}{z^4 - 1} = \frac{g(z)}{h(z)}$$

where $g(z) = z^2$ and $h(z) = z^4 - 1$.

Here we have $g(i) = i^2 = -1 \neq 0$

And $h(i) = i^4 - 1 = 1 - 1 = 0$.

Also, $h'(z) = 4z^3$ and so $h'(i) = 4i^3 = -4i \neq 0$

So, $g(i) \neq 0$, $h(i) = 0$, $h'(i) \neq 0$.

By a thm from class

$z_0 = i$ is a simple pole of f

and

$$\operatorname{Res}(f, i) = \frac{g(i)}{h'(i)} = \frac{i^2}{-4} = \frac{-1}{-4i}$$

$$= \frac{1}{4} \cdot \frac{1}{i} = \frac{1}{4}(-i) = \frac{-i}{4}$$

① (g) If $z \neq 0$ but near 0, then

$$f(z) = \left(\frac{\cos(z)-1}{z}\right)^2 = \left(\frac{-1+\cos(z)}{z}\right)^2$$

$$= \left(\frac{-1+1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots}{z}\right)^2$$

$$= \left(\frac{-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots}{z}\right)^2$$

$$= \left(-\frac{z}{2!}+\frac{z^3}{4!}-\frac{z^5}{6!}+\cdots\right)^2$$

$$= \frac{1}{4}z^2 - \frac{1}{24}z^4 + \cdots$$

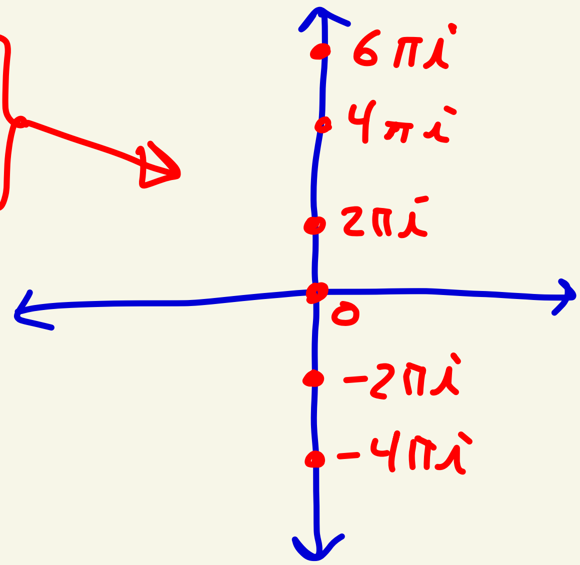
So we have a removable singularity at $z_0 = 0$. And

$$\mathrm{Res}(f;0) = 0$$

(2) The singular points of $f(z) = \dfrac{1}{e^z - 1}$

are when $e^z - 1 = 0$ or $e^z = 1$.

These are located at $z = 2\pi i k$ where $k \in \mathbb{Z}$



Here we have

$$f(z) = \frac{g(z)}{h(z)}$$

where $g(z) = 1$, $h(z) = e^z - 1$. And, $h'(z) = e^z$.

And $g(2\pi i k) = 1 \neq 0$,

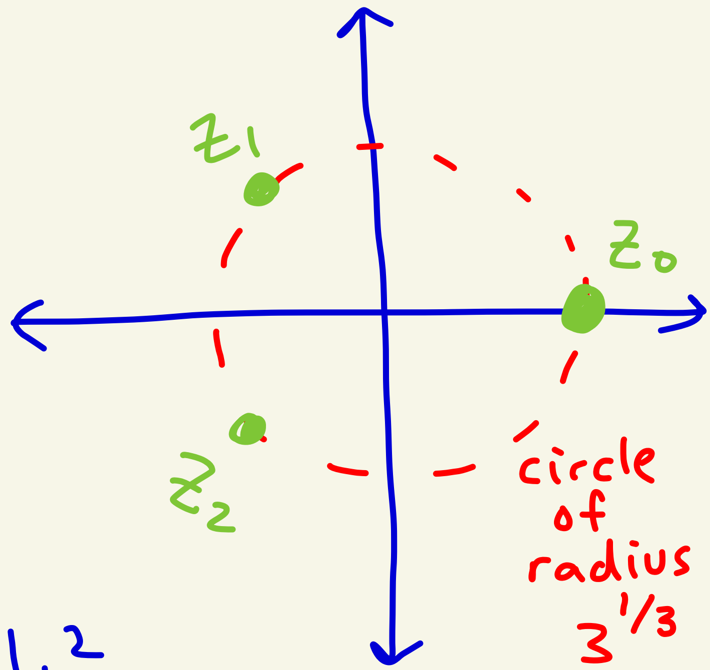
$h(2\pi i k) = e^{2\pi i k} - 1$, $h'(2\pi i k) = e^{2\pi i k} = 1 \neq 0$.

So, by a thm in class these are all simple poles and

$$\mathrm{Res}(f; 2\pi i k) = \frac{g(2\pi i k)}{h'(2\pi i k)} = \frac{1}{1} = 1.$$

③ The singular points of $f(z) = \dfrac{1}{z^3 - 3}$

are when $z^3 - 3 = 0$.



Let's solve this:

$$z^3 = 3 = 3 \cdot e^{0i}$$

Solutions are:

$$z_k = 3^{1/3} e^{\left(\frac{0}{3} + \frac{2\pi}{3}k\right)i}$$

$$= 3^{1/3} e^{\frac{2\pi}{3}ki}, \quad k = 0, 1, 2$$

$$= \underbrace{3^{1/3}}_{z_0}, \ \underbrace{3^{1/3} e^{\frac{2\pi}{3}i}}_{z_1}, \ \underbrace{3^{1/3} e^{\frac{4\pi}{3}i}}_{z_2}$$

So, z_0, z_1, z_2 are the singularities of $f(z)$.

Let $g(z) = 1$, $h(z) = z^3 - 3$.

Then, $f(z) = \dfrac{g(z)}{h(z)}$.

And $h'(z) = 3z^2$.

Note that

$$g(z_0) = 1 \neq 0$$
$$h(z_0) = 0$$
$$h'(z_0) = 3(3^{1/3})^2 = 3 \cdot 3^{2/3} \neq 0$$

Thus, $z_0 = 3^{1/3}$ is a simple pole and

$$\operatorname{Res}(f; 3^{1/3}) = \frac{g(3^{1/3})}{h'(3^{1/3})} = \frac{1}{3 \cdot 3^{2/3}}$$

Also, $g(z_1) = 1 \neq 0$

$$h(z_1) = 0$$
$$h'(z_1) = 3\left(3^{1/3} e^{\frac{2\pi}{3}i}\right)^2 = 3 \cdot 3^{2/3} e^{\frac{4\pi}{3}i} \neq 0$$

So, z_1 is a simple pole and

$$\operatorname{Res}(f; z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{3 \cdot 3^{2/3} e^{\frac{4\pi}{3}i}}$$

We also have

$$g(z_2) \neq 0$$

$$h(z_2) = 0$$

$$h'(z_2) = 3\left(3^{1/3} e^{\frac{4\pi}{3}i}\right)^2 = 3 \cdot 3^{2/3} e^{\frac{8\pi}{3}i}$$

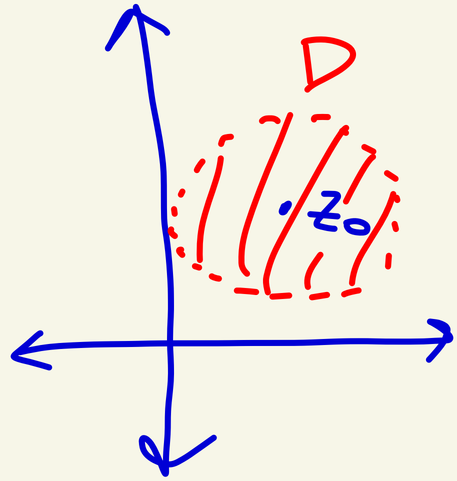
$$= 3 \cdot 3^{2/3} e^{\frac{2\pi}{3}i} \neq 0$$

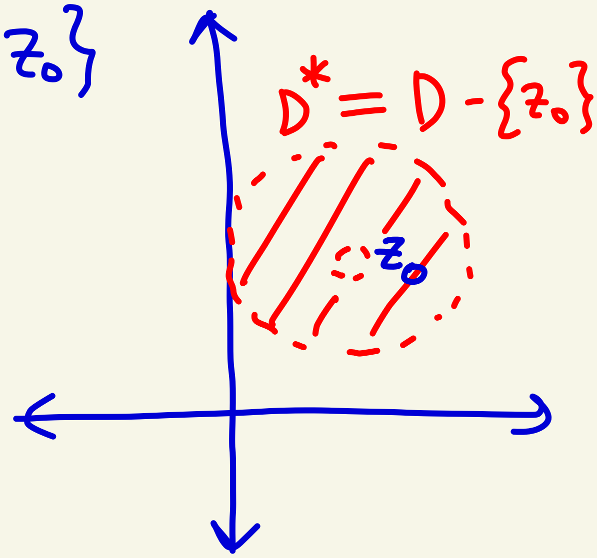
So, we have a simple pole at z_2 and

$$\operatorname{Res}(f; z_2) = \frac{g(z_2)}{h'(z_2)}$$

$$= \frac{1}{3 \cdot 3^{2/3} \cdot e^{\frac{2\pi}{3}i}}$$

④ Since f_1 and f_2 both have simple poles at z_0 we know that there exists a disc D around z_0, and two functions $\varphi_1(z)$ and $\varphi_2(z)$ that are analytic in D, $\varphi_1(z_0) \neq 0$, $\varphi_2(z_0) \neq 0$ and for all $z \in D^* = D - \{z_0\}$ we have





$$f_1(z) = \frac{\varphi_1(z)}{z - z_0} \quad \text{and}$$

$$f_2(z) = \frac{\varphi_2(z)}{z - z_0}.$$

Thus, for $z \in D^*$ we have

$$(f_1 \cdot f_2)(z) = \frac{\varphi_1(z)\varphi_2(z)}{(z - z_0)^2}$$

where $\varphi_1(z) \cdot \varphi_2(z)$ is analytic in D, ie at $z = z_0$, and $\varphi_1(z_0) \cdot \varphi_2(z_0) \neq 0$.

Thus, we have a pole of 2 at $z = z_0$ and so,

$$\mathrm{Res}(f; z_0) = \frac{\left([\varphi_1(z)\varphi_2(z)]' \right)\Big|_{z=z_0}}{1!}$$

$$= \varphi_1'(z_0)\,\varphi_2(z_0) + \varphi_1(z_0)\,\varphi_2'(z_0)$$