# Math 5680
# HW 7 Solutions

① Let $z_1, z_2 \in A$.


Suppose that $f$ is analytic on $A$ and $f'(z_1) \neq 0$.

Suppose $D = D(z_2; r)$ is contained in $A$.

We will show that $f$ is not constant on $D$.

Suppose that $f(z) = c$ for all $z \in D$ where $c$ is a fixed complex number.

Let $f_c : A \to \mathbb{C}$ be the constant function $f_c(z) = c$ for all $z \in A$.

Then $f_c$ is also analytic on $A$ and $f(z) = c = f_c(z)$ for all $z \in D$.

By the identity theorem,

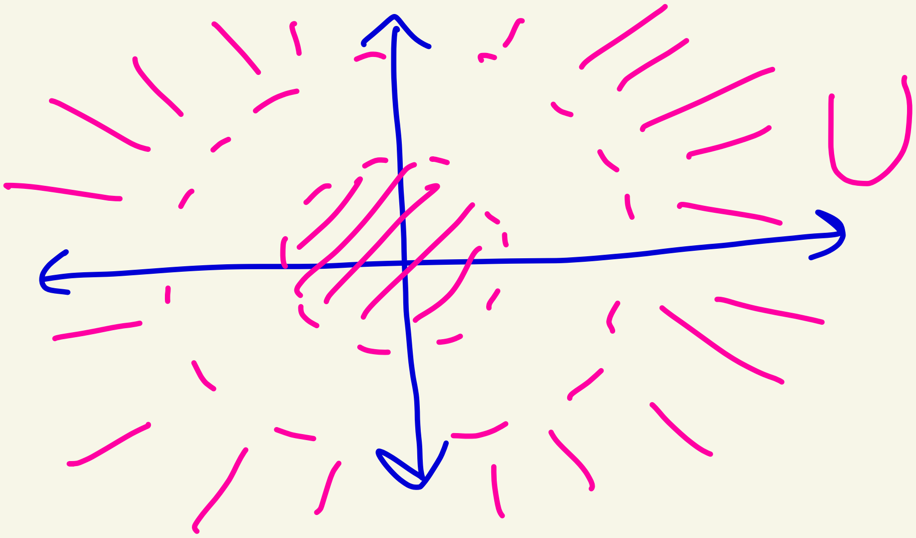
$f(z) = f_c(z)$ for all $z \in A$.

Thus, $f(z) = c$ for all $z \in A$.

But then $f'(z) = 0$ for all $z \in A$.

This contradicts $f'(z_1) \neq 0$.

Thus, $f$ cannot be constant in any disc surrounding $z_2$ contained in $A$ $\blacksquare$

② 


Suppose that $g: \mathbb{C} \to \mathbb{C}$ is an entire function where $g(z) = f(z)$ for all $z \in U$.

Let $h(z) = z$ for all $z \in \mathbb{C}$.
Then for $|z| < 1$ we have $z \in U$ and thus $g(z) = f(z) = z = h(z)$.
So $g$ and $h$ are analytic functions on all of $\mathbb{C}$ [which is a region] and are equal in the neighborhood $D(0;1) = \{z \mid |z| < 1\} \subseteq \mathbb{C}$.

By the identity theorem, $g(z) = h(z)$ for all $z \in \mathbb{C}$.

Thus, $g(z) = z$ for all $z \in \mathbb{C}$.

But we also have $g(z) = f(z) = z^2$ for all $z$ with $|z| > 2$

Thus, for all $z$ with $|z| > 2$ we have
$$z = g(z) = z^2.$$

However this isn't true for say $z = 3$ which has $|z| > 2$ since
$$3 = z \neq z^2 = 9.$$

Therefore, there is no entire function $g$ with $g(z) = f(z)$ for all $z \in U$.