Math 5800

10/13/21

Math 5800 - ~~Spring~~ Fall 2021 - Test 1

- You can only use your mind to take this exam. No help from any sources or people. No books, no notes, no online, etc.
- Use blank paper (like printer paper) if you have it please.
- On the first page of your exam, before any of your solutions, put these three things:
  (a) Your name.
  (b) The time period that you chose.
  (c) Copy this statement and then sign your signature after it:

  "Everything on this test is my own work. I did not use any sources or talk to anyone about this exam." **your signature**
- After your name and the above statement with signature, start putting your solutions to the problems. Please put them in order. That is, first problem 1, then problem 2, etc. You can put each one on its own page.
- Please scan your test using a scanner (such as a free one on your phone) and put it into one pdf document with your problems in order.
- To get a clean scan, make sure there is plenty of light, the phone is held flat directly above the paper, and the paper is placed on a flat object such as the floor or a table.
- Please upload your answer to canvas.

This is the cover page for Test 1 on Monday

Test 1 Monday

Mon 8am → Tues 12 noon

You pick 2.5 hour window

Canvas will time you once
you start

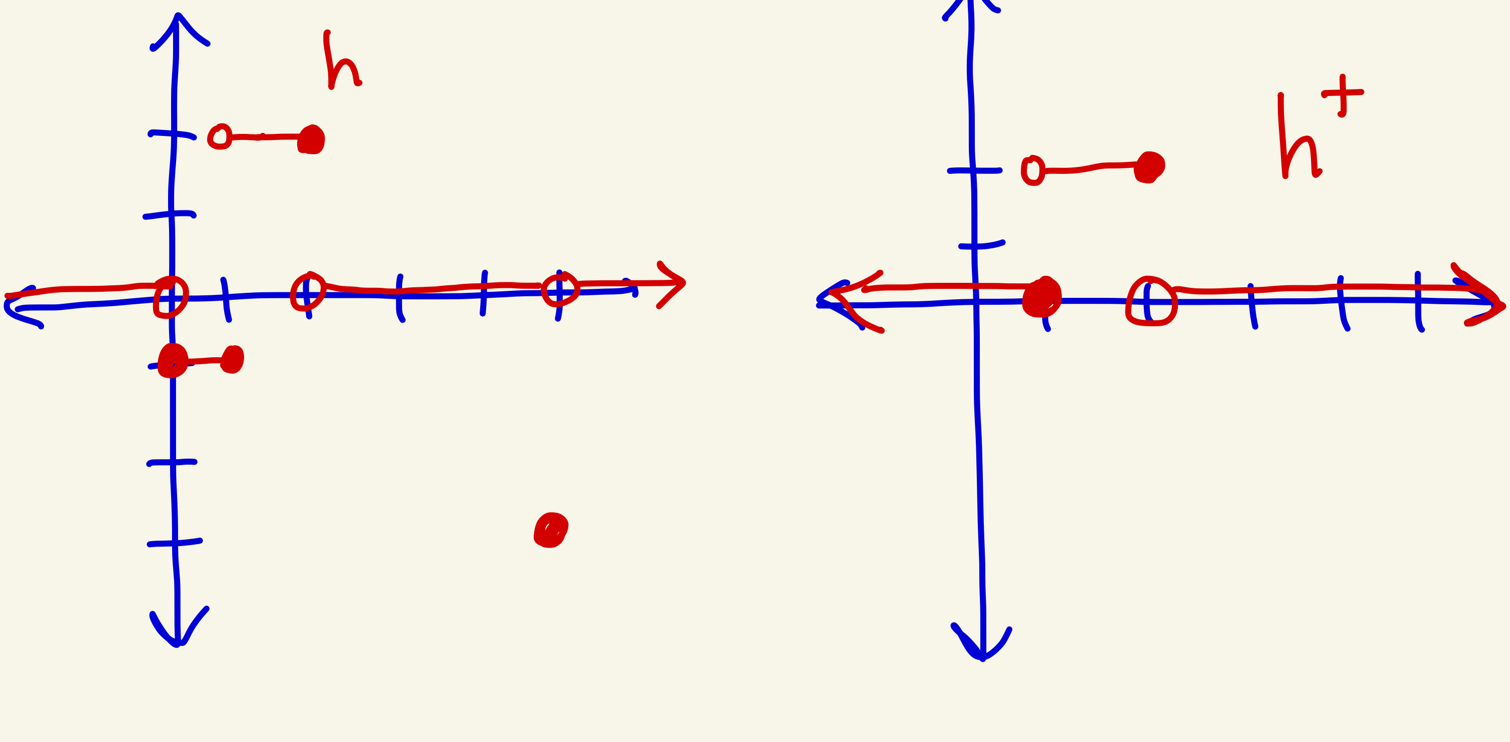
Continue from last time...

Def: Let $h : \mathbb{R} \to \mathbb{R}$

Define $h^+ : \mathbb{R} \to \mathbb{R}$ as

$$h^+(x) = \begin{cases} h(x) & \text{if } h(x) \geq 0 \\ 0 & \text{if } h(x) < 0 \end{cases}$$

Ex: $h = 2 \cdot \chi_{(1,2]} - 3 \cdot \chi_{[5,5]} - 1 \cdot \chi_{[0,1]}$



<u>Theorem:</u> If h is a step function, then h^+ is a step function.

<u>proof:</u> HW 4 $\boxtimes$

<u>Lemma:</u> Let $f, g \in L^0$.
Suppose $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ are non-decreasing sequences of step functions where $\varphi_n \to f$ almost everywhere and $\psi_n \to g$ almost everywhere. Suppose $\lim_{n\to\infty} \int \varphi_n$ and $\lim_{n\to\infty} \int \psi_n$ exist.
Suppose $f \geq g$ almost everywhere
[ie $f(x) \geq g(x)$ for almost all x]
Then, $\lim_{n\to\infty} \int \varphi_n \geq \lim_{n\to\infty} \int \psi_n$

<u>proof:</u> Fix some integer $m \geq 1$.

Consider the sequence of step functions $(\psi_m - \varphi_n)_{n=1}^{\infty}$,

that is

$$\psi_m - \varphi_1 \;,\; \psi_m - \varphi_2 \;,\; \psi_m - \varphi_3 \;,\; \cdots$$

Note that

$$(\psi_m - \varphi_n)(x) = \psi_m(x) - \varphi_n(x)$$
$$\geq \psi_m(x) - \varphi_{n+1}(x)$$
$$= (\psi_m - \varphi_{n+1})(x)$$

for all $n \geq 1$ and $x \in \mathbb{R}$.

$(\varphi_n)_{n=1}^{\infty}$ is non-decreasing

$\varphi_n(x) \leq \varphi_{n+1}(x)$

$-\varphi_n(x) \geq -\varphi_{n+1}(x)$

So, $(\psi_m - \varphi_n)_{n=1}^{\infty}$ is non-increasing [in n].

Let

$$S_1 = \{ x \mid \lim_{n\to\infty} \varphi_n(x) \neq f(x) \}$$

$$S_2 = \{ x \mid \underbrace{f(x) \not\geq g(x)}_{\text{ie } f(x) < g(x)} \}$$

$$S_3 = \{ x \mid \lim_{m\to\infty} \psi_m(x) \neq g(x) \}$$

By assumption, S_1, S_2, S_3 all have measure zero.

Let $S = S_1 \cup S_2 \cup S_3$.

Then S has measure zero.

And,

$$\mathbb{R} - S = \{ x \mid \lim_{n\to\infty} \varphi_n(x) = f(x),\ f(x) \geq g(x), \text{ and } \lim_{m\to\infty} \psi_m(x) = g(x) \}$$

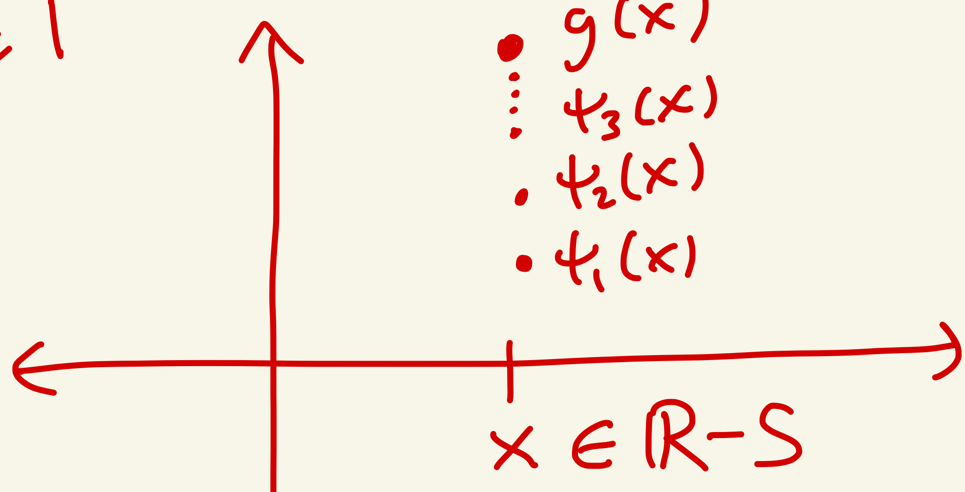
$\mathbb{R} - S$ is an almost everywhere set.

Let $x \in \mathbb{R}-S$.

Then,

$$\lim_{n\to\infty}\left(\psi_m(x)-\varphi_n(x)\right) = \psi_m(x)-f(x) \overset{f(x)\geq g(x)}{\leq} \psi_m(x)-g(x) \leq 0$$

We know $\psi_m(x)-g(x)\leq 0$ because $\lim_{m\to\infty}\psi_m(x)=g(x)$ and $(\psi_m(x))_{m=1}^{\infty}$ is non-decreasing. So, $\psi_m(x)\leq g(x)$ for all $m\geq 1$



Thus, $(\psi_m - \varphi_n)_{n=1}^{\infty}$ is a

non-increasing sequence of step functions where

$\lim_{n\to\infty} (\psi_m(x) - \varphi_n(x)) \le 0$ for all $x \in \mathbb{R} - S$ [ie almost all x].

Consider the sequence $((\psi_m - \varphi_n)^+)_{n=1}^{\infty}$

that is,

$(\psi_m - \varphi_1)^+, (\psi_m - \varphi_2)^+, \dots$

This is a non-increasing sequence of step functions which are all non-negative.

Since $\lim_{n\to\infty} (\psi_m(x) - \varphi_n(x)) \le 0$

for all $x \in \mathbb{R} - S$, we know that

that $\lim_{n\to\infty} (\psi_m(x) - \varphi_n(x))^+ = 0$.

for all $x \in \mathbb{R} - S$.

Thus, $((\psi_m - \varphi_n)^+)_{n=1}^{\infty}$ is a non-increasing non-negative sequence of step functions where $\lim_{n\to\infty} (\psi_m - \varphi_n)^+(x) = 0$

for almost all x.

By Monday's lemma,

$$\lim_{n\to\infty} \int (\psi_m - \varphi_n)^+ = 0$$

But $(\psi_m - \varphi_n)(x) \le (\psi_m - \varphi_n)^+(x)$

for all x.

Thus,

$$\lim_{n\to\infty} \int (\psi_m - \varphi_n) \le \lim_{n\to\infty} \int (\psi_m - \varphi_n)^+ = 0$$

So, $\lim_{n\to\infty} \int (\psi_m - \varphi_n) \le 0.$

Thus, for any fixed $m \geq 1$ we have

$$\int \psi_m - \lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int (\psi_m - \varphi_n) \leq 0$$

Now take the limit as $m \to \infty$ to get

$$\lim_{m\to\infty} \int \psi_m - \lim_{n\to\infty} \int \varphi_n \leq 0$$

So,

$$\lim_{m\to\infty} \int \psi_m \leq \lim_{n\to\infty} \int \varphi_n$$

$\blacksquare$

<u>Theorem:</u> Let $f \in L^0$.
Then, $\int f$ is well-defined.

<u>pf:</u> Suppose $(\varphi_n)_{n=1}^{\infty}$ and $(\psi_m)_{m=1}^{\infty}$ are two non-decreasing sequences of step functions with $\varphi_n \to f$ almost everywhere and $\psi_m \to f$ almost everywhere and $\lim_{n\to\infty} \int \varphi_n$ exists and $\lim_{n\to\infty} \int \psi_m$ exists. If you apply the previous lemma twice using $g = f$ we get $\lim_{n\to\infty} \int \varphi_n \leq \lim_{m\to\infty} \int \psi_m$

and $\lim_{m\to\infty} \int \psi_m \leq \lim_{n\to\infty} \int \varphi_n$.

Hence $\lim_{m\to\infty} \int \psi_m = \lim_{n\to\infty} \int \varphi_n$.

So, $\int f = \lim_{n\to\infty} \int \psi_m$ and

$\int f = \lim_{n\to\infty} \int \varphi_n$ are equal

and $\int f$ is well-defined. $\blacksquare$

<u>Corollary</u>: Let $f, g \in L^0$.

If $f(x) \geq g(x)$ for almost all x, then $\int f \geq \int g$.

<u>proof</u>: This is the previous lemma before the theorem. $\blacksquare$