Math 5800

10/20/21

On Monday, 10/25 I have to miss class. I will record a lesson for that day and email to everyone and post it on canvas.

please watch the recording before Weds, 10/27

I'll probably record it tomorrow or the next day.

No class on Monday, just watch recording please.

Recap:

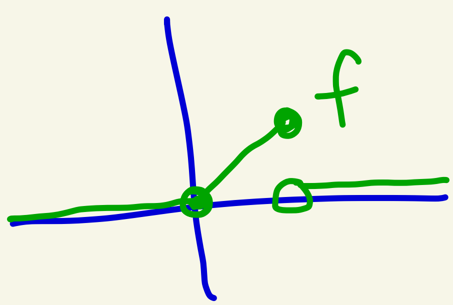
- $f : \mathbb{R} \to \mathbb{R}$

  $f \in L^0$ means there exists a non-decreasing sequence of step functions $(\varphi_n)_{n=1}^{\infty}$ where $\varphi_n \to f$ almost everywhere and $\lim_{n\to\infty} \int \varphi_n$ exists.

  We defined $\int f = \lim_{n\to\infty} \int \varphi_n$

- All step functions are in L^0

- $f \in L^0$ where $f(x) = \begin{cases} x & \text{when } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$



- We showed that $\int f$ for $f \in L^0$ is well-defined

- We showed that if $f, g \in L^0$ and $f(x) \geq g(x)$ for almost all x then $\int f \geq \int g$

Theorem: Let $f, g \in L^0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$ and $\beta \geq 0$, then $\alpha f + \beta g \in L^0$ and

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

proof: HW. $\blacksquare$

Note: One can show that L^0 is not a linear space. That is, one can find $f, g \in L^0$ where $f - g \notin L^0$.

See WJ book on pg 54-55 or Weir pg. 43.

I'll email this to you.

This motivates us to enlarge our space of functions.

<u>Def:</u> We define the space L^1 of <u>Lebesgue interable functions</u> as

$$L^1 = \left\{ f \;\middle|\; \begin{array}{l} f = g - h \text{ where} \\ g, h \in L^0 \end{array} \right\}$$

Given $f = g - h$ with $g, h \in L^0$ we define

$$\int f = \underbrace{\int g}\; - \underbrace{\int h}$$

L^0 integrals

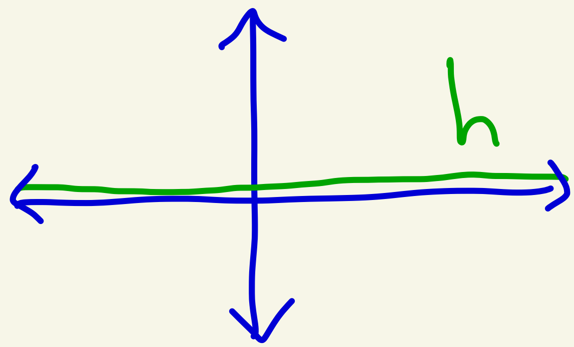
We say that $f : \mathbb{R} \to \mathbb{R}$ is <u>integrable</u> if $f \in L^1$.

Theorem: $L^0 \subseteq L^1$

proof: Let $g \in L^0$.

Set $h : \mathbb{R} \to \mathbb{R}$ where $h(x) = 0$ for all $x \in \mathbb{R}$.

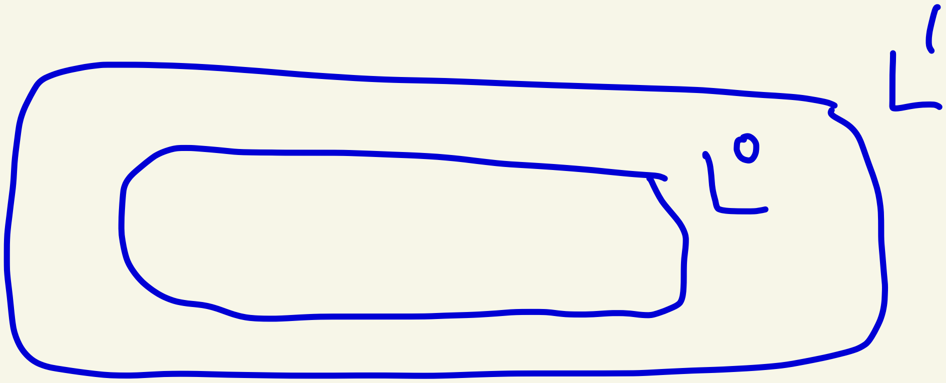
Note that $h = \chi_{(0,0)} = \chi_{\emptyset}$



So, $h \in L^0$ since h is a step function [by HW].

Then, $g = \underbrace{g}_{\text{in } L^0} - \underbrace{h}_{\text{in } L^0} \in L^1$.

∎



Theorem: [WJ book Corollary 1.5.1]

The integral defined on L^1 is well-defined. That is, suppose that $f \in L^1$ where $f = g - h$ and $f = v - w$ where $g, h, v, w \in L^0$.

Then $\underbrace{\int g - \int h}_{\int f} = \underbrace{\int v - \int w}_{\int f}$

Proof: We have that

$$g - h = f = v - w.$$

So, $g + w = v + h$.

By HW [pg 4 in notes] we have that $g + w \in L^0$ and $v + h \in L^0$

and $\int g+w = \int g + \int w$

and $\int v+h = \int v + \int h$.

Thus,

$$\int g + \int w = \int g+w$$

$g+w$
$= v+h$

$$= \int v+h = \int v + \int h.$$

Hence, $\int g + \int w = \int v + \int h$

Therefore, $\int g - \int h = \int v - \int w$

$\blacksquare$

Theorem [WJ book - Thm 1.5.2]

L^1 is a linear space of functions. That is, if $f, u \in L^1$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta u \in L^1$

Furthermore, if this is the case then

$$\int (\alpha f + \beta u) = \alpha \int f + \beta \int u$$

proof:

Since $f \in L^1$ and $u \in L^1$ we know that there exist $g, h, v, w \in L^0$ where

$f = g - h$ and $u = v - w$.

Case 1: Suppose $\alpha \geq 0$ and $\beta \geq 0$

Note that

$$\alpha f + \beta u = \alpha(g-h) + \beta(v-w)$$
$$= (\alpha g + \beta v) - (\alpha h + \beta w)$$

By previous thm / HW problem since $\alpha, \beta \geq 0$ and $g, v \in L^{\circ}$ we know that $\alpha g + \beta v \in L^{\circ}$ and

$$\int (\alpha g + \beta v) = \alpha \int g + \beta \int v$$

Since $\alpha, \beta \geq 0$ and $h, w \in L^{\circ}$ we know that $\alpha h + \beta w \in L^{\circ}$ and

$$\int (\alpha h + \beta w) = \alpha \int h + \beta \int w$$

Thus,

$$\alpha f+\beta u = \underbrace{(\alpha g+\beta v)}_{\text{in } L^0} - \underbrace{(\alpha h+\beta w)}_{\text{in } L^0} \in L^1$$

And

$$\int(\alpha f+\beta u) = \int(\alpha g+\beta v) - \int(\alpha h+\beta w)$$

previous page

$$\overset{\downarrow}{=} \alpha\int g+\beta\int v - \left[\alpha\int h+\beta\int w\right]$$

$$= \alpha\left[\int g-\int h\right]+\beta\left[\int v-\int w\right]$$

$$\boxed{\begin{aligned} f&=g-h \\ u&=v-w \end{aligned}}$$

$$\overset{\downarrow}{=} \alpha\int f+\beta\int u$$

This concludes case 1.

The remaining cases are proved in the same way as case 1 using the given decompositions below.

case 2: $\alpha < 0, \beta < 0$

Write

$$\alpha f + \beta u = \alpha g - \alpha h + \beta v - \beta w$$
$$= ((-\alpha)h + (-\beta)w) - ((-\alpha)g + (-\beta)v)$$

case 3: $\alpha \geq 0, \beta < 0$

Write

$$\alpha f + \beta u = \alpha g - \alpha h + \beta v - \beta w$$
$$= (\alpha g + (-\beta)w) - (\alpha h + (-\beta)v)$$

case 4: $\alpha < 0, \beta \geq 0$

Write

$$\alpha f + \beta u = \alpha g - \alpha h + \beta v - \beta w$$
$$= ((-\alpha)h + \beta v) - ((-\alpha)g + \beta w)$$

$\blacksquare$