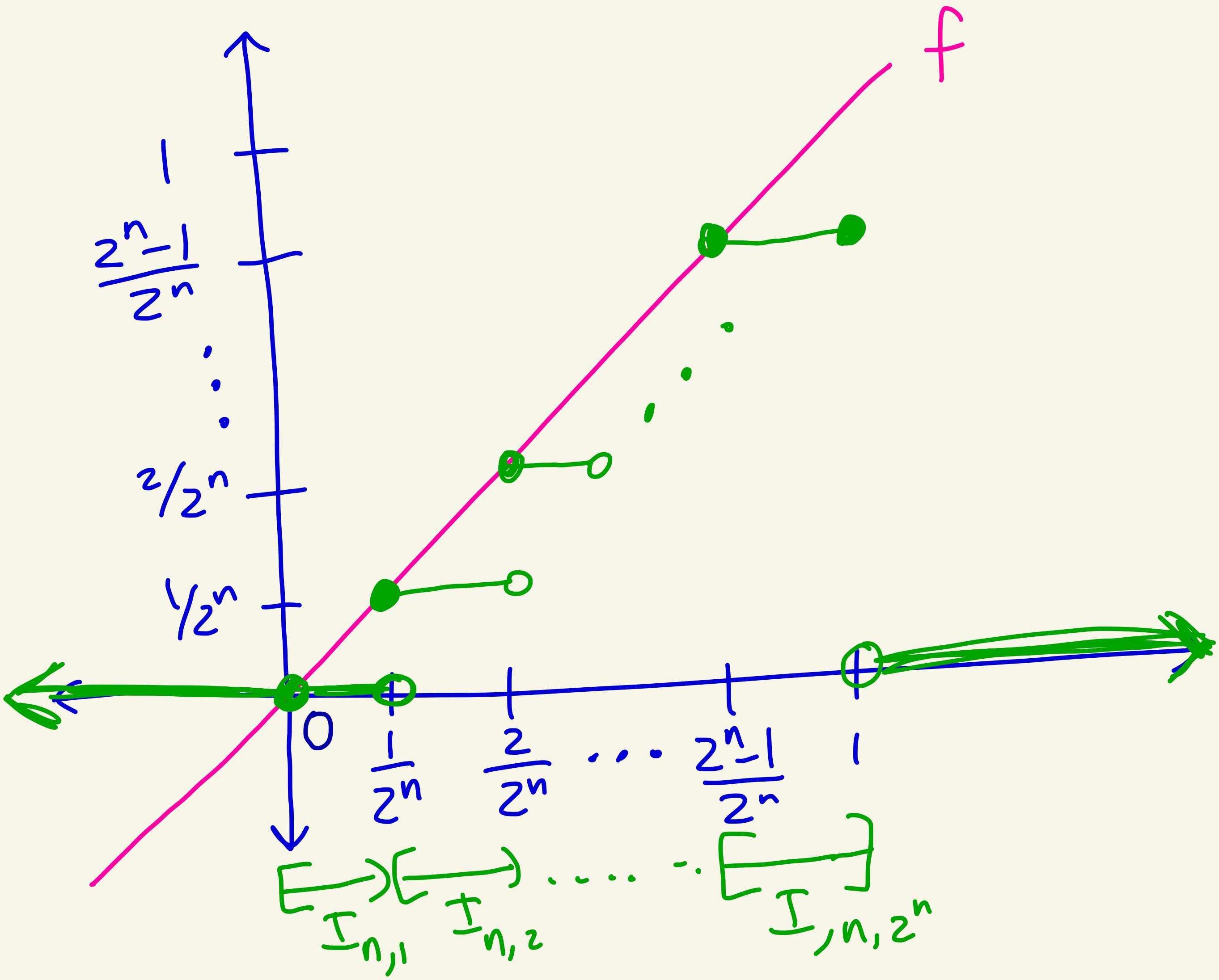
Math 5800

10/4/21

Ex: Last time we had

$f(x) = x$ for all $x \in \mathbb{R}$.

We constructed the standard construction for f on $[0,1]$



$$\varphi_n = 0 \cdot \chi_{I_{n,1}} + \frac{1}{2^n} \cdot \chi_{I_{n,2}} + \dots + \frac{2^n-1}{2^n} \chi_{I_{n,2^n}}$$

Claim: $\gamma_n \to f$ pointwise on $[0,1]$.

Proof of claim:

Let $x \in [0,1]$.

We want to show $\lim_{n\to\infty} \gamma_n(x) = f(x)$.

Part 1: One has that

$$|\gamma_n(x) - f(x)| \leq \frac{1}{2^n}$$

Proof of part 1:

We break this into two cases.

Suppose first that $0 \leq x < 1$.

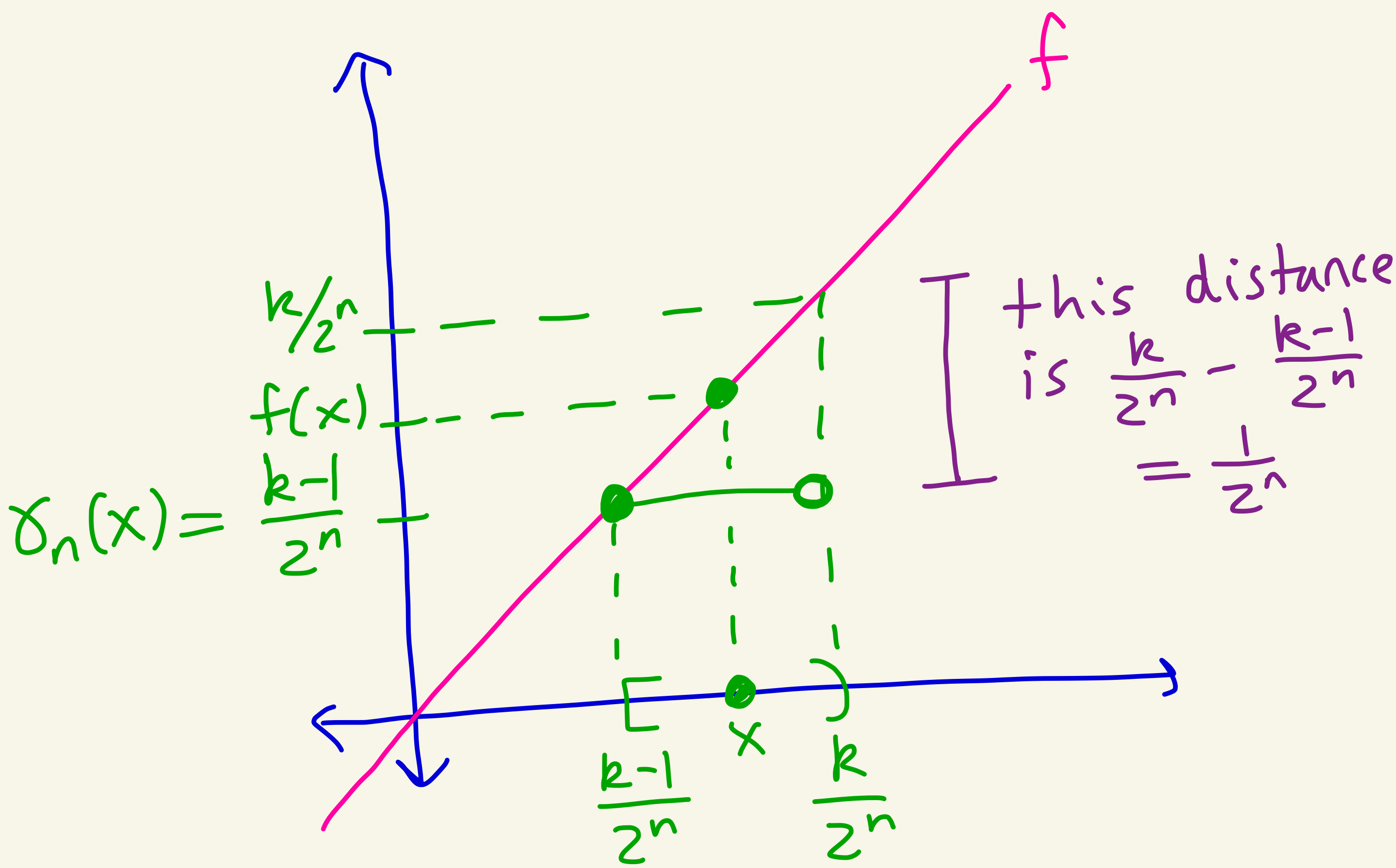
Then $x \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ where $k = 1, 2, \ldots, 2^n$

Since f is an increasing function

$$\gamma_n(x) = \inf\{f(t) \mid t \in I_{n,k}\}$$

$$= f\left(\frac{k-1}{2^n}\right) = \frac{k-1}{2^n}$$

$\boxed{\text{f of left-endpoint of } \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \text{ since } f \text{ is an increasing function}}$



From the picture we see that at most

$$|\gamma_n(x) - f(x)| < \frac{1}{2^n}$$

Now suppose $x = 1$.

Then, $\gamma_n(x) = \gamma_n(1) = \frac{2^n - 1}{2^n}$

So,

$$|\gamma_n(1) - f(1)| = \left|\frac{2^n-1}{2^n} - 1\right|$$

$$= \left|-\frac{1}{2^n}\right| = \frac{1}{2^n}.$$

So, if $0 \leq x \leq 1$,

then $|\gamma_n(x) - f(x)| \leq \frac{1}{2^n}$.

$\boxed{\text{Part 1}}$

Part 2: $\lim_{n\to\infty} \gamma_n(x) = f(x)$

Recall we are assuming $0 \leq x \leq 1$

pf of part 2: Let $\varepsilon > 0$.

We know that $|\gamma_n(x) - f(x)| \leq \frac{1}{2^n}$.

And, $\frac{1}{2^n} < \varepsilon$

iff $\frac{1}{\varepsilon} < 2^n$

iff $\log_2\left(\frac{1}{\varepsilon}\right) < n$.

Set $N > \log_2\left(\frac{1}{\varepsilon}\right)$.

Then if $n \geq N$, then $n > \log_2\left(\frac{1}{\varepsilon}\right)$ and we will have

$$|\gamma_n(x) - f(x)| \leq \frac{1}{2^n} < \varepsilon.$$

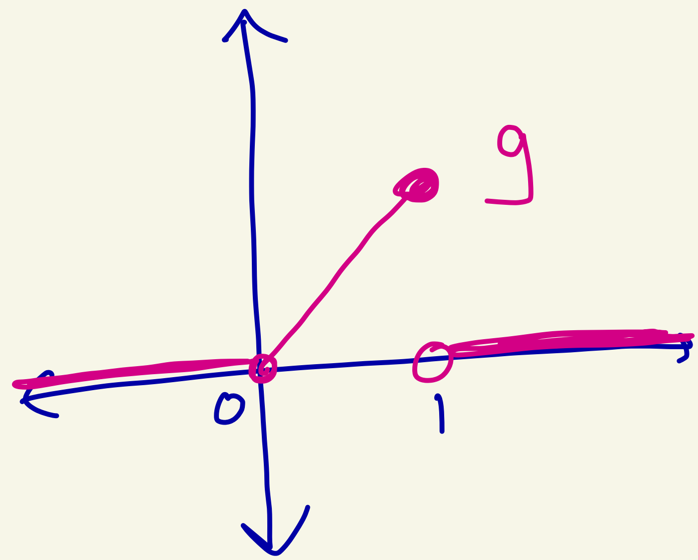
$\boxed{\text{Part 2}}$

$\boxed{\text{Claim}}$

But $\gamma_n \not\to f$ outside of $[0,1]$. But we can modify f.

<u>Ex:</u> Let

$$g(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$



Then, g has the same standard construction γ_n as f does on $[0,1]$ because $f(x) = g(x)$ for all $x \in [0,1]$.

And we just saw that $\gamma_n(x) \to g(x)$ when $x \in [0,1]$.

If $x \notin [0,1]$, $\gamma_n(x) = 0 = g(x)$ $\forall n \geq 1$ and so $\gamma_n(x) \to g(x)$

Thus, $\gamma_n \to g$ pointwise on all of $\mathbb{R}$. $\blacksquare$

<u>Theorem:</u> Let $f : \mathbb{R} \to \mathbb{R}$ be bounded on $[a,b]$. Let $(\gamma_n)_{n=1}^{\infty}$ be the standard construction for f on $[a,b]$.

Then:

① $(\gamma_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions

② $\gamma_n(x) \leq f(x)$ for all $n \geq 1$ and $x \in [a,b]$

<u>proof:</u>

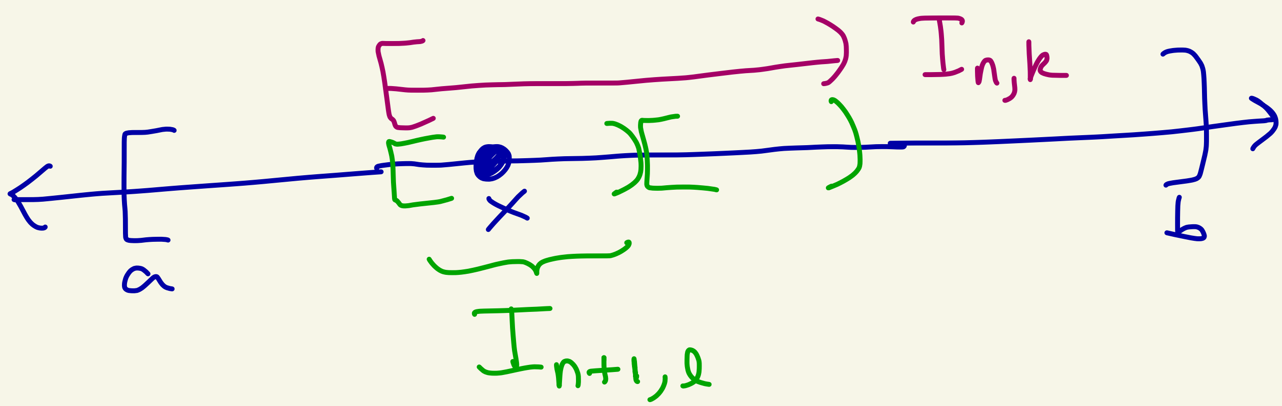
① Let $x \in [a,b]$ and $n \geq 1$.

Then, $\gamma_n(x) = \sum_{i=1}^{2^n} m_{n,i} \cdot \chi_{I_{n,i}}(x)$

and $\gamma_{n+1}(x) = \sum_{j=1}^{2^{n+1}} m_{n+1,j} \cdot \chi_{I_{n+1,j}}(x)$

Then, $x \in I_{n,k}$ for some $1 \le k \le 2^n$ and also $x \in I_{n+1,\ell}$ for some $1 \le \ell \le 2^{n+1}$.

And, $I_{n+1,\ell} \subseteq I_{n,k}$ because at each stage n, we subdivide each interval in half to get to the $(n+1)$-stage



Thus,

$$\{f(t) \mid t \in I_{n+1,\ell}\} \subseteq \{f(t) \mid t \in I_{n,k}\}$$

$I_{n+1,\ell} \subseteq I_{n,k}$

So,

$$\gamma_{n+1}(x) = m_{n+1,\ell}$$
$$= \inf\{f(t) \mid t \in I_{n+1,\ell}\}$$
$$\geq \inf\{f(t) \mid t \in I_{n,k}\}$$

$A \subseteq B$
$\inf(A) \geq \inf(B)$

$$= m_{n,k} = \gamma_n(x).$$

$\boxed{1}$

② Let $x \in [a,b]$ and $n \geq 1$.

Let $I_{n,k}$ be the subinterval that x is in with $1 \leq k \leq 2^n$.

Then,

$$\gamma_n(x) = m_{n,k}$$

$$= \inf\{f(t) \mid t \in I_{n,k}\}$$

$$\leq f(x)$$

because $x \in I_{n,k}$

$\boxed{②}$

Def: Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions defined on $S \subseteq \mathbb{R}$.
So, $f_n : S \to \mathbb{R}$ for all $n \geq 1$.
Let $f : S \to \mathbb{R}$.
We say that <u>f_n converges to f almost everywhere in S</u> if the following are true:

① there exists $A \subseteq S$ where
$$\lim_{n \to \infty} f_n(x) = f(x)$$
for all $x \in A$.

and ② $S - A$ has measure zero.

If $S = \mathbb{R}$ in the previous definition and 1 and 2 are true then we are saying that $f_n \to f$ on some almost everywhere set $A \subseteq \mathbb{R}$.

In this special case we just say that <u>f_n converges to f almost everywhere</u>

instead of saying "f_n converges to f almost everywhere in $\mathbb{R}$"