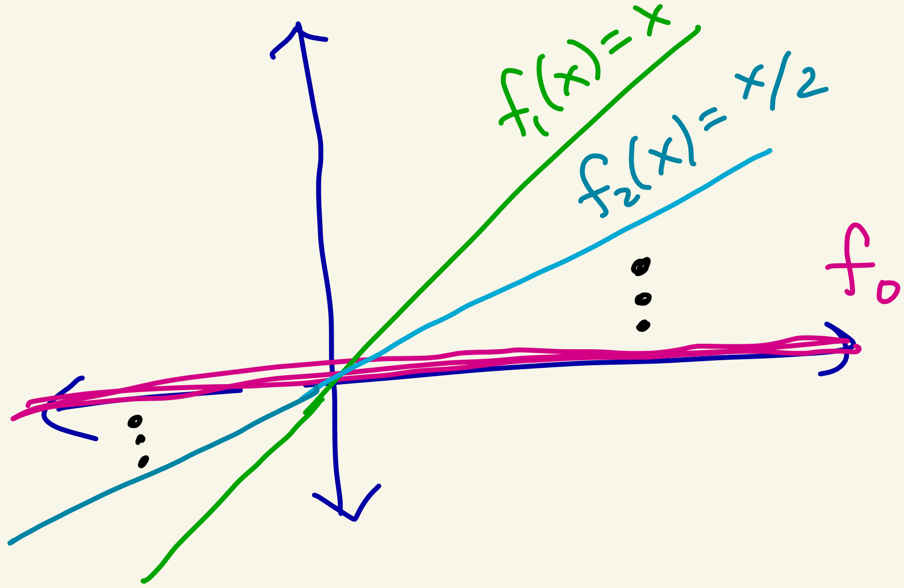
Math 5800

10/6/21

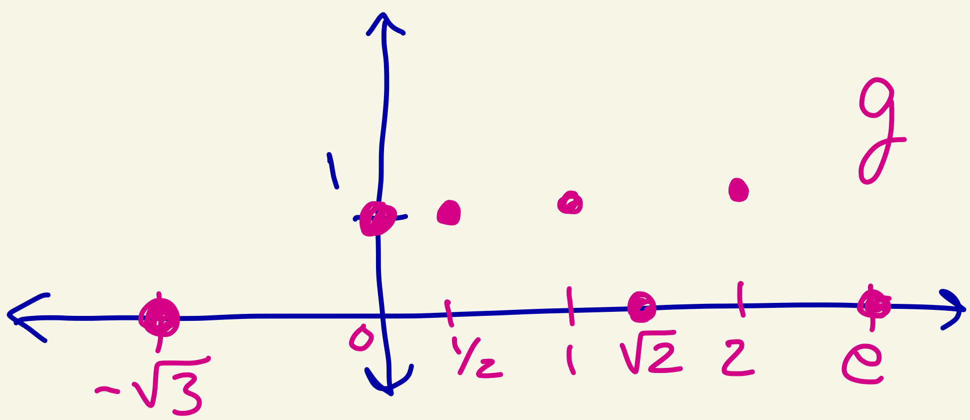
Ex: Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{n}$.

We saw that $f_n \to f_0$ on all of $\mathbb{R}$ where $f_0(x) = 0 \ \forall x \in \mathbb{R}$



Let $g : \mathbb{R} \to \mathbb{R}$ be

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$



Here $f_n \to g$ on $\mathbb{R} - \mathbb{Q}$

and $\mathbb{Q}$ has measure zero.

So, $f_n \to g$ on almost all of $\mathbb{R}$.

Or, $f_n \to g$ almost everywhere.

Note: Let $(\varphi_n)_{n=1}^{\infty}$ be a non-decreasing sequence of step functions where $\left(\int \varphi_n\right)_{n=1}^{\infty}$

is a convergent sequence.

or equivalently, as we saw, that $\left(\int \varphi_n\right)_{n=1}^{\infty}$ is bounded

Let

$$A = \left\{ x \in \mathbb{R} \;\middle|\; (\varphi_n(x))_{n=1}^{\infty} \text{ converges} \right\}$$

We showed that $\mathbb{R} \sim A$ has measure zero.

So A is an almost everywhere set.

Let $f: \mathbb{R} \to \mathbb{R}$ be any function where $f(x) = \lim_{n\to\infty} \varphi_n(x)$

for all $x \in A$.

So, $f(x)$ can be anything if $x \notin A$.

Then,

$$\varphi_n \to f$$

pointwise on A.

So, $\varphi_n \to f$

almost everywhere because

$\mathbb{R} - A$ has measure zero.

For example,

$$f(x) = \begin{cases} \lim_{n\to\infty} \varphi_n(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Note

Def: [Def 1.5.1 in WJ]

Let L^0 denote the set of functions $f: \mathbb{R} \to \mathbb{R}$ such that

1. there exists a non-decreasing sequence of step functions $(\varphi_n)_{n=1}^{\infty}$ that converges almost everywhere to f.

and

2. $\lim_{n \to \infty} \int \varphi_n$ converges

[equivalent to $\left(\int \varphi_n\right)_{n=1}^{\infty}$ bounded]

We define the integral for such an f as

$$\int f = \lim_{n \to \infty} \int \varphi_n$$

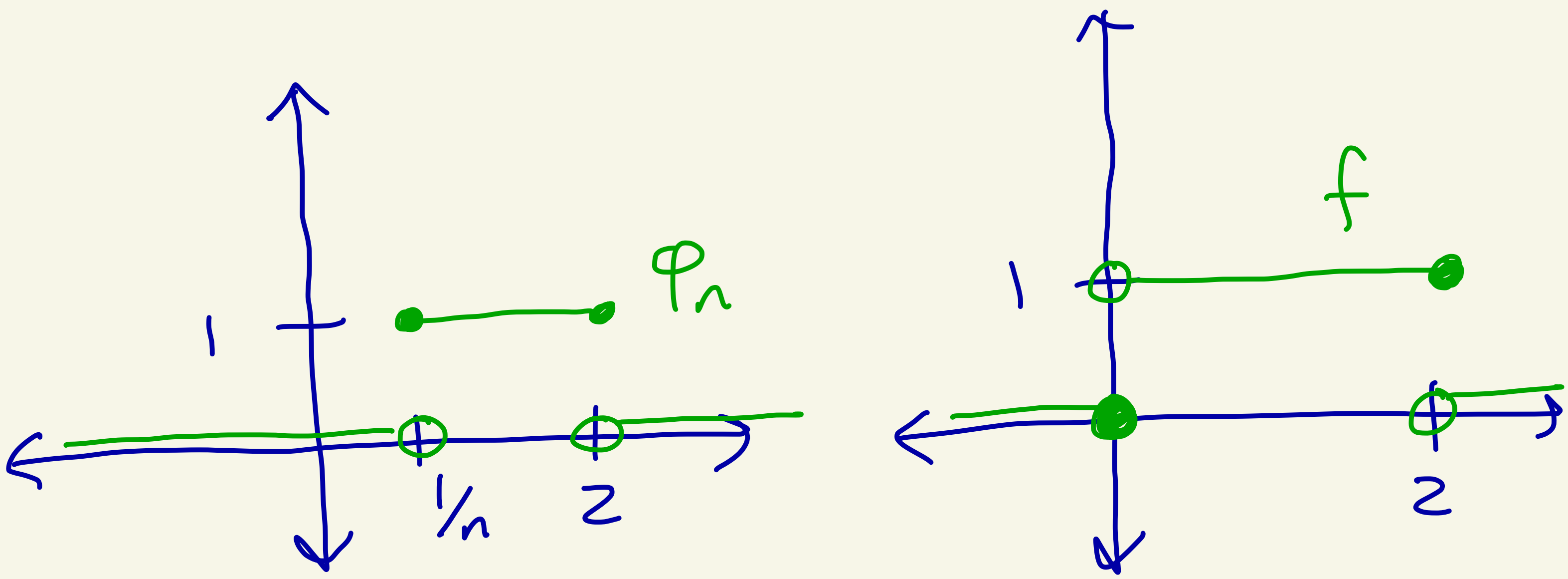
In Weir, L° is denoted by L^{inc}.

In Haaser / Sullivan its denoted by $\tilde{\mathcal{J}}$.

Ex: Let

$$f(x) = \begin{cases} 1 & \text{if } x \in (0,2] \\ 0 & \text{if } x \notin (0,2] \end{cases}$$

Let $\varphi_n = \chi_{[\frac{1}{n}, 2]}$



We showed previously that $\varphi_n \to f$ on all of $\mathbb{R}$. So, $\varphi_n \to f$ almost everywhere on $\mathbb{R}$.

We showed $(\varphi_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions with $\lim_{n\to\infty} \int \varphi_n = 2$.

So, $f \in L^0$.

And, $\int f = \lim_{n\to\infty} \int \varphi_n = 2$. $\blacksquare$

Note: $f = \chi_{(0,2]}$ is a step function.

We could have used $\varphi_n = \chi_{(0,2]}$.

So our non-decreasing sequence would be

$$\chi_{(0,2]}, \chi_{(0,2]}, \chi_{(0,2]}, \ldots$$

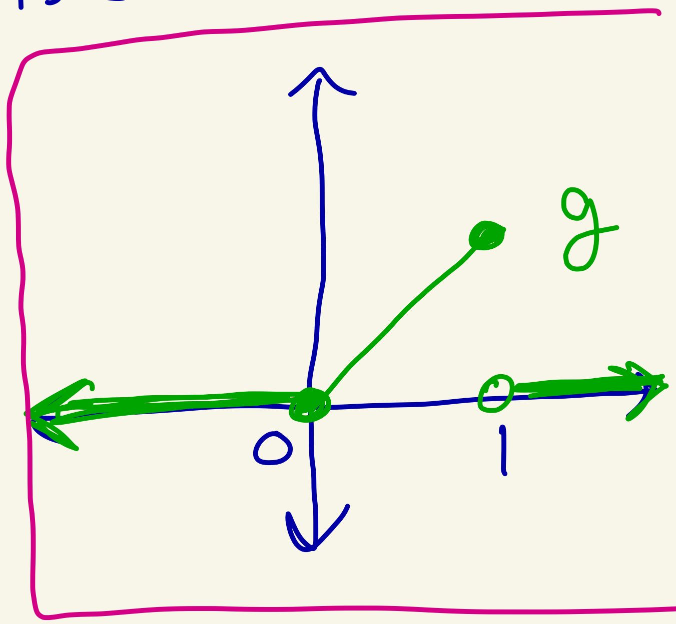
This converges to f everywhere and so $f \in L^0$ and

$$\int f = \lim_{n\to\infty} \int \varphi_n = \lim_{n\to\infty} \int \chi_{(0,2]}$$

$$= \lim_{n\to\infty} 2 = 2 \quad \blacksquare$$

Ex: Let

$$g(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$



Let's show that $g \in L^0$ and $\int g = \frac{1}{2}$.

Let $(\gamma_n)_{n=1}^{\infty}$ be the standard construction for g on $[0,1]$. We know that

1. $(\gamma_n)_{n=1}^{\infty}$ is a non-decreasing sequence of step functions

and 2. $\gamma_n \to g$ on all of $\mathbb{R}$ [and hence almost everywhere]

Let's show that $\lim_{n\to\infty} \int \gamma_n$ exists. $\boxed{\text{pg } 10}$

Recall

$$\gamma_n = 0 \cdot \chi_{[0,\frac{1}{2^n})} + \frac{1}{2^n} \cdot \chi_{[\frac{1}{2^n},\frac{2}{2^n})} + \frac{2}{2^n} \cdot \chi_{[\frac{2}{2^n},\frac{3}{2^n})} + \dots + \frac{2^n-1}{2^n} \chi_{[\frac{2^n-1}{2^n},1]}$$

Then,

$$\int \gamma_n = 0 \cdot \ell\left(\left[0,\tfrac{1}{2^n}\right)\right) + \frac{1}{2^n} \cdot \ell\left(\left[\tfrac{1}{2^n},\tfrac{2}{2^n}\right)\right) + \frac{2}{2^n} \cdot \ell\left(\left[\tfrac{2}{2^n},\tfrac{3}{2^n}\right)\right) + \dots + \frac{2^n-1}{2^n} \ell\left(\left[\tfrac{2^n-1}{2^n},1\right]\right)$$

$$= \underbrace{0 \cdot \frac{1}{2^n}}_{0} + \frac{1}{2^n} \cdot \frac{1}{2^n} + \frac{2}{2^n} \cdot \frac{1}{2^n} + \dots + \frac{2^n-1}{2^n} \cdot \frac{1}{2^n}$$

$$= \frac{1}{2^n \cdot 2^n}\left[1+2+3+\dots+(2^n-1)\right]$$

pg 11

$$= \frac{1}{2^n \cdot 2^n}\left[1+2+3+\cdots+(2^n-1)\right]$$

$$= \frac{1}{2^n \cdot 2^n}\left[\frac{(2^n-1)(2^n-1+1)}{2}\right]$$

$$\boxed{m = 2^n-1}$$

$$\boxed{1+2+3+\ldots+m = \frac{m(m+1)}{2}}$$

$$= \frac{1}{2^n \cdot 2^n} \cdot \frac{(2^n-1)(2^n)}{2} = \frac{2^n-1}{2 \cdot 2^n}$$

So, $\boxed{\int \gamma_n = \frac{2^n-1}{2 \cdot 2^n}}$

Thus, $\lim_{n\to\infty} \int \gamma_n = \lim_{n\to\infty} \frac{1-\frac{1}{2^n}}{2}$

$\boxed{\text{divide top/bottom by } 2^n}$

$$= \frac{1-0}{2} = \frac{1}{2}$$

Thus, $g \in L^{\circ}$ and

$$\int g = \lim_{n\to\infty} \int \gamma_n = \frac{1}{2}.$$

$\blacksquare$

HW: If f is a step function then $f \in L^{\circ}$