Math 5800

11/10/21

HW typos

HW 7 #11(a) – I had the graphs labelled g_1, g_1, g_1. It should be g_1, g_2, g_3. I fixed this in the solutions.

HW 7 #10 – On the second page of the solution I had $\chi_I(x) = 0$ and $h(x) = 1$ when $x \in \mathbb{Q} \cap I$. It should be $\chi_I(x) = 1$ and $h(x) = 0$. I fixed this in the solutions.

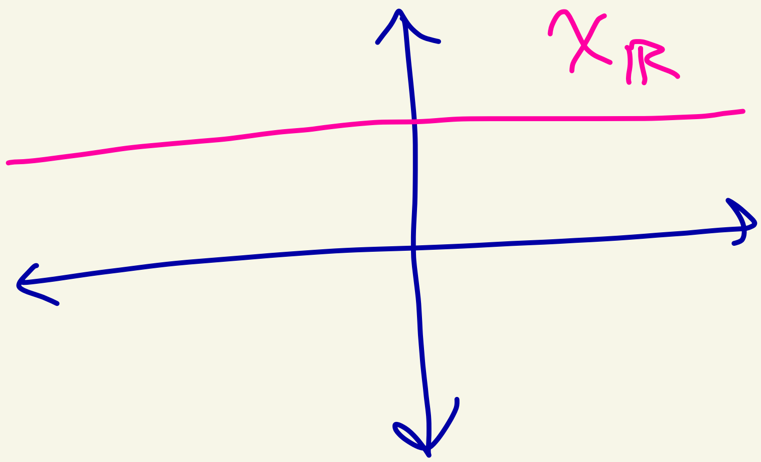
HW 6 #6 – In the problem statement I put $f_n \to g$ almost everywhere. It should be $f_n \to f$ almost everywhere. It's fixed now.

HW 6 - 4(b)

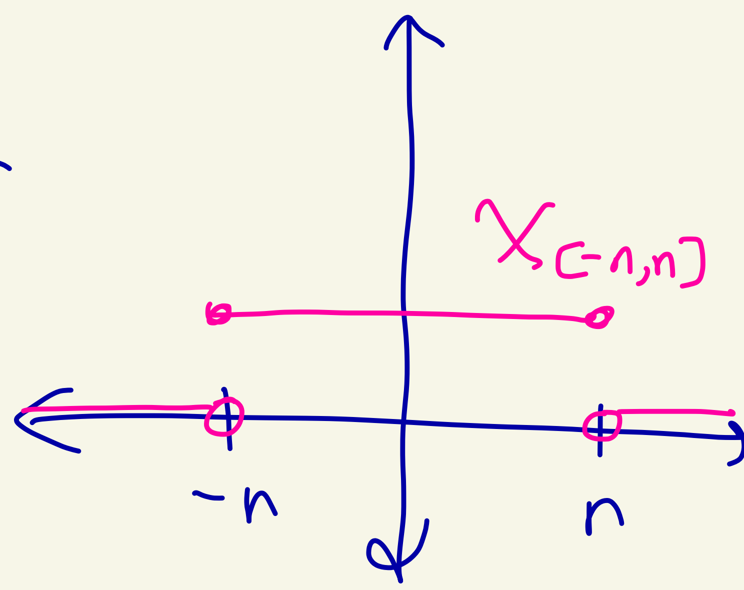
When calculating $\gamma_n(1)$ I forgot to square the value. Its fixed online now.

Idea from last time is that $\mathrm{mid}\{-g, f, g\}$ "truncates" f by g and $-g$. $[g \geq 0]$

<u>Ex:</u> From HW, $\chi_{\mathbb{R}}$ is not in L^1.



But $\chi_{[-n,n]}$ is in L^1 for each $n \geq 1$ and $\chi_{[-n,n]} \to \chi_{\mathbb{R}}$ as $n \to \infty$ on all of $\mathbb{R}$.



Def: Let $f: \mathbb{R} \to \mathbb{R}$.

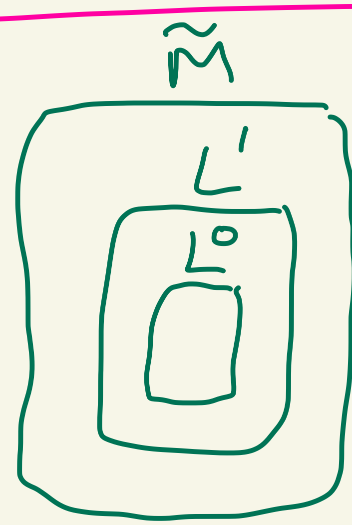
We say that f is measurable if for every $g \in L^1$ where g is non-negative (ie $g \geq 0$) we have that $\text{mid}\{-g, f, g\} \in L^1$.

We will denote the space of measurable functions by $\tilde{M}$.

Idea: f is measurable means that if we truncate f by an integrable [integrable means L^1] $g \geq 0$ then that truncation $\text{mid}\{-g, f, g\}$ will be integrable.

HW 9 #5 - Let $h, k \in L^1$.
Then, $\min\{h,k\}$ and $\max\{h,k\}$ are in L^1. Here

$$\max\{h,k\}(x) = \max\{h(x), k(x)\}$$
$$\min\{h,k\}(x) = \min\{h(x), k(x)\}$$



Theorem: $L^1 \subseteq \tilde{M}$

That is, if f is integrable, then f is measurable.

proof: Let $f \in L^1$.

Pick some $g \in L^1$ with $g \geq 0$.

We need to show that

$\text{mid}\{-g, f, g\}$ is in L^1.

By Hw 9 #5, $\min\{f,g\} \in L^1$.

Thus again by Hw 9 #5

$$\max\{-g, \min\{f,g\}\} \in L^1.$$

Thus,

$$\mathrm{mid}\{-g, f, g\} = \max\{-g, \min\{f, g\}\}$$

is in L^1.

Thus, f is measurable. $\blacksquare$

<u>Theorem</u> (Dominated Convergence theorem)

Let $(f_n)_{n=1}^{\infty}$ be a sequence of L^1 functions. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f_n \to f$ almost everywhere. Suppose that there exists $g \in L^1$, $g \geq 0$, where $|f_n| \leq g$ for all $n \geq 1$.

[means: $|f_n(x)| \leq g(x)$ for all $x \in \mathbb{R}$ and $n \geq 1$]

Then, $f \in L^1$ and

$$\int f = \lim_{n \to \infty} \int f_n$$

<u>Proof:</u> Handout. $\blacksquare$

Theorem: Let $f:\mathbb{R}\to\mathbb{R}$.
f is measurable iff there exists a sequence $(f_n)_{n=1}^{\infty}$ of L^1 functions such that $f(x)=\lim_{n\to\infty} f_n(x)$ for almost all $x\in\mathbb{R}$.

proof: Let $f:\mathbb{R}\to\mathbb{R}$.

($\Leftarrow$) Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of L^1 functions and

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for almost all $x\in\mathbb{R}$.

Let's show that $f\in\tilde{M}$.

Pick any $g \in L^1$ with $g \geq 0$.

We want to show that

$\text{mid}\{-g, f, g\}$ is in L^1.

Let $h_n = \text{mid}\{-g, f_n, g\}$ for $n \geq 1$.

Then since $-g, f_n, g \in L^1$ we know that

$$h_n = \text{mid}\{-g, f_n, g\} = \max\{-g, \min\{f_n, g\}\}$$

is in L^1.

Note that

$$|h_n(x)| = \begin{cases} |-g(x)| & \text{if } f_n(x) < -g(x) \\ |f_n(x)| & \text{if } -g(x) \leq f_n(x) \leq g(x) \\ |g(x)| & \text{if } g(x) < f_n(x) \end{cases}$$

We have $|-g(x)| = g(x)$ ← $\boxed{g \geq 0}$

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Also, when $-g(x) \leq f_n(x) \leq g(x)$

then $|f_n(x)| \leq g(x)$.

And, $|g(x)| = g(x)$.

Thus, $|h_n(x)| \leq g(x)$ for all x and $n \geq 1$.

HW 9 #7(b) - $(f_n)_{n=1}^{\infty}$ is a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $g \geq 0$, $f : \mathbb{R} \to \mathbb{R}$ with $f_n \to f$ almost everywhere.

Then, $\lim\limits_{n \to \infty} \text{mid}\{-g, f_n, g\}(x) = \text{mid}\{-g, f, g\}(x)$ for almost all x

By HW 9 #7(b),

$$\lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} \text{mid}\{-g, f_n, g\}(x)$$

$$= \text{mid}\{-g, f, g\}(x)$$

for almost all x.

Thus, $(h_n)_{n=1}^{\infty} = (\mathrm{mid}\{-g, f_n, g\})_{n=1}^{\infty}$ is a sequence of L^1 functions that converges almost everywhere to $\mathrm{mid}\{-g, f, g\}$ and

$$|h_n(x)| = |\mathrm{mid}\{-g, f_n, g\}(x)|$$

$$\leq g(x) \quad \text{[where } g \in L^1\text{]}$$

for all $x \in \mathbb{R}$ and $n \geq 1$.

Thus, by the dominated convergence theorem, $\mathrm{mid}\{-g, f, g\} \in L^1$.

So, $f \in \tilde{M}$.