Math 5800

11/17/21

We continue from last time

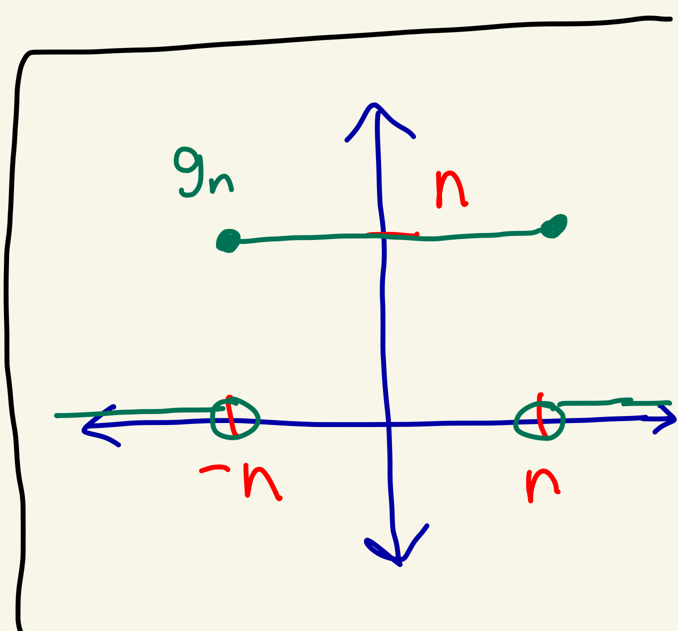
Theorem: Let $f:\mathbb{R}\to\mathbb{R}$. f is measurable iff there exists a sequence $(f_n)_{n=1}^{\infty}$ of L^1 functions where $f_n \to f$ almost everywhere.

proof:

($\Leftarrow$) We did this last week.

($\Rightarrow$) Suppose that f is measurable.

Let
$$g_n = n\cdot \chi_{[-n,n]}$$
for $n \geq 1$



Note, $g_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $n \geq 1$.

So, $g_n \geq 0$, ie g_n is non-negative.

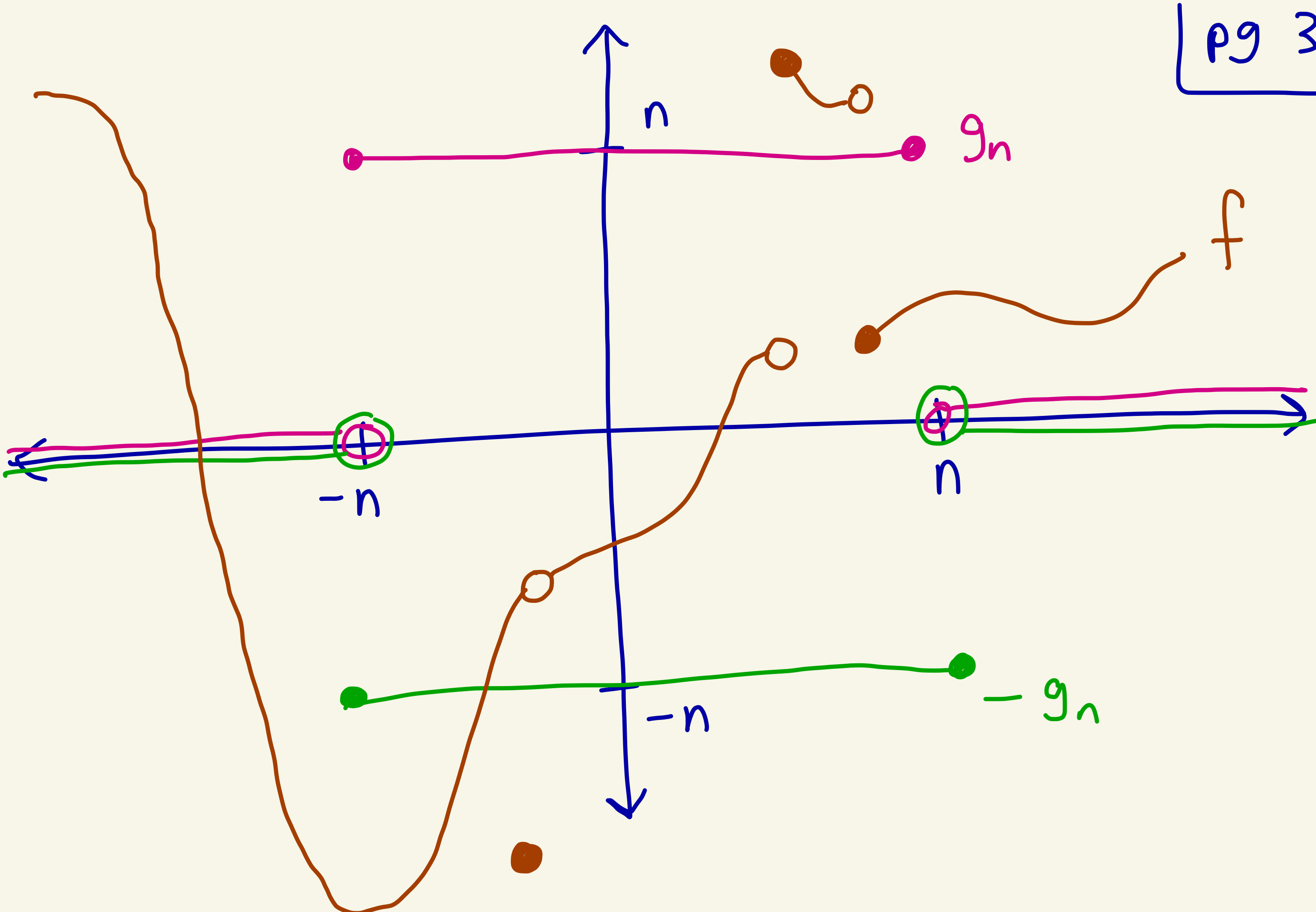
Also, $g_n \in L^1$ because

$g_n = n \cdot \chi_{[-n,n]}$ is a step function.

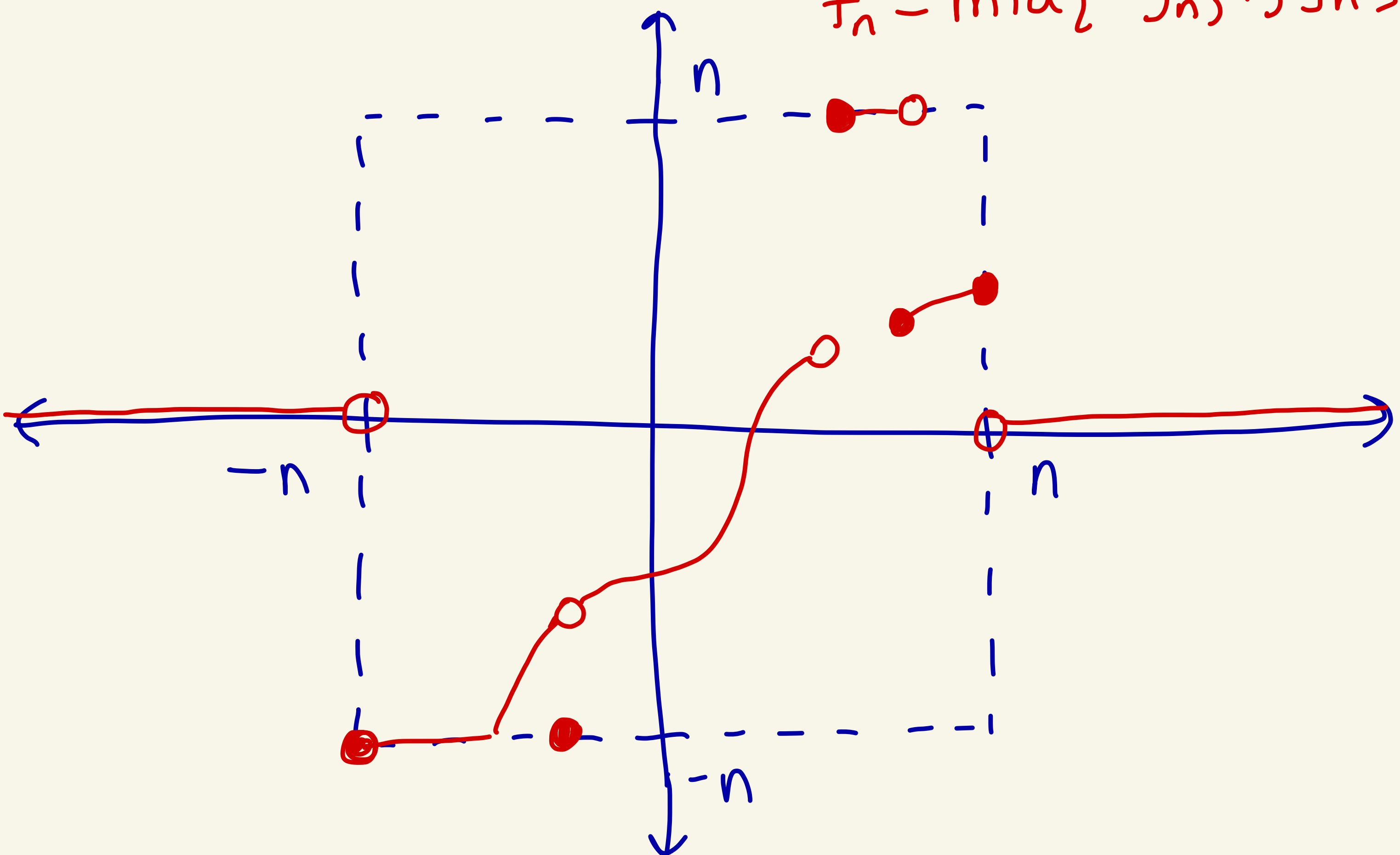
Let

$$f_n = \text{mid}\{-g_n, f, g_n\}$$

Let's draw a picture.



$$f_n = \mathrm{mid}\{-g_n, f, g_n\}$$



So, $f_n = \text{mid}\{-g_n, f, g_n\}$ truncates f into a $2n \times 2n$ box centered at the origin.

Claim 1: $f_n \to f$ on all of $\mathbb{R}$

pf of claim 1: This is HW 9 #2. (We don't even need f to be measurable for claim 1.)

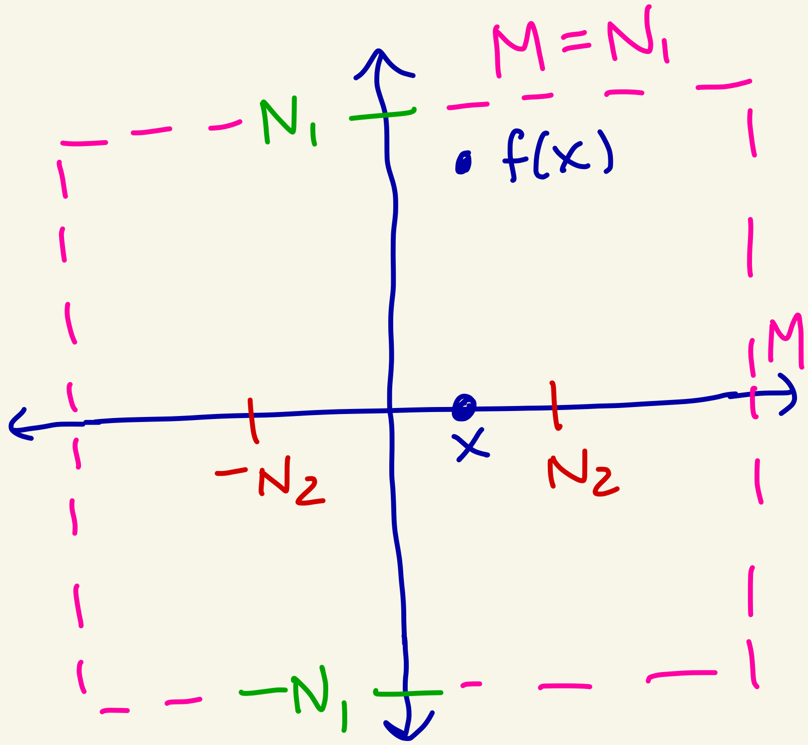
Fix some $x \in \mathbb{R}$.

We will show $\lim_{n\to\infty} f_n(x) = f(x)$.

Pick $N_1 > 0$ large enough so that $-N_1 \le f(x) \le N_1$

Pick $N_2 > 0$ large enough so that $-N_2 \le x \le N_2$.

Set $M = \max\{N_1, N_2\}$



Thus,

$$-M \le f(x) \le M \quad \text{and} \quad -M \le x \le M.$$

So,

$$-g_M(x) = -M \cdot \overbrace{\chi_{[-M,M]}(x)}^{1}$$

$$= -M \le f(x) \le M$$

$$= M \cdot \underbrace{\chi_{[-M,M]}}_{1}(x) = g_M(x).$$

That is, $\underbrace{-g_M(x) \le f(x) \le g_M(x).}$

So,

$$f_M(x) = \text{mid}\{-g_M(x), f(x), g_M(x)\} = f(x).$$

Note that if $n \ge M$, then $x \in [-M, M] \subseteq [-n, n]$ and so

$$\chi_{[-M,M]}(x) = 1 = \chi_{[-n,n]}(x).$$

Thus, if $n \geq M$ then

$$-g_n(x) = -n \cdot \underbrace{\chi_{[-n,n]}(x)}_{1}$$

$$= -n \leq -M = -M \cdot \underbrace{\chi_{[-M,M]}(x)}_{1}$$

$$= -g_M(x) \leq f(x) \leq g_M(x)$$

$$= M \cdot \underbrace{\chi_{[-M,M]}(x)}_{1}$$

$$= M \leq n = n \cdot \underbrace{\chi_{[-n,n]}(x)}_{1}$$

$$= g_n(x).$$

Therefore if $n \geq M$, then

$$-g_n(x) \leq f(x) \leq g_n(x).$$

So, if $n \geq M$ then

$$f_n(x) = \text{mid}\{-g_n, f, g_n\}(x)$$
$$= \text{mid}\{-g_n(x), f(x), g_n(x)\}$$
$$= f(x)$$

$$\boxed{-g_n(x) \leq f(x) \leq g_n(x)}$$

Thus if $\varepsilon > 0$ and $n \geq M$ we have

$$|f_n(x) - f(x)| = |f(x) - f(x)|$$
$$= 0 < \varepsilon$$

Thus, $\lim_{n \to \infty} f_n(x) = f(x)$.

Since x was arbitrary, $f_n \to f$ on all of $\mathbb{R}$.

$\boxed{\text{claim } 1}$

Claim 2: $f_n \in L^1$ for $n \geq 1$

proof of claim 2:

Because f is measurable and $g_n \in L^1$ and $g_n \geq 0$ we know by def of measurable that $\text{mid}\{-g_n, f, g_n\}$ is in L^1.

Thus, $f_n = \text{mid}\{-g_n, f, g_n\} \in L^1$

$\boxed{\text{claim 2}}$

By claim 1 and claim 2, $(f_n)_{n=1}^{\infty}$ is a sequence of L^1 functions with $f_n \to f$ on all of $\mathbb{R}$.

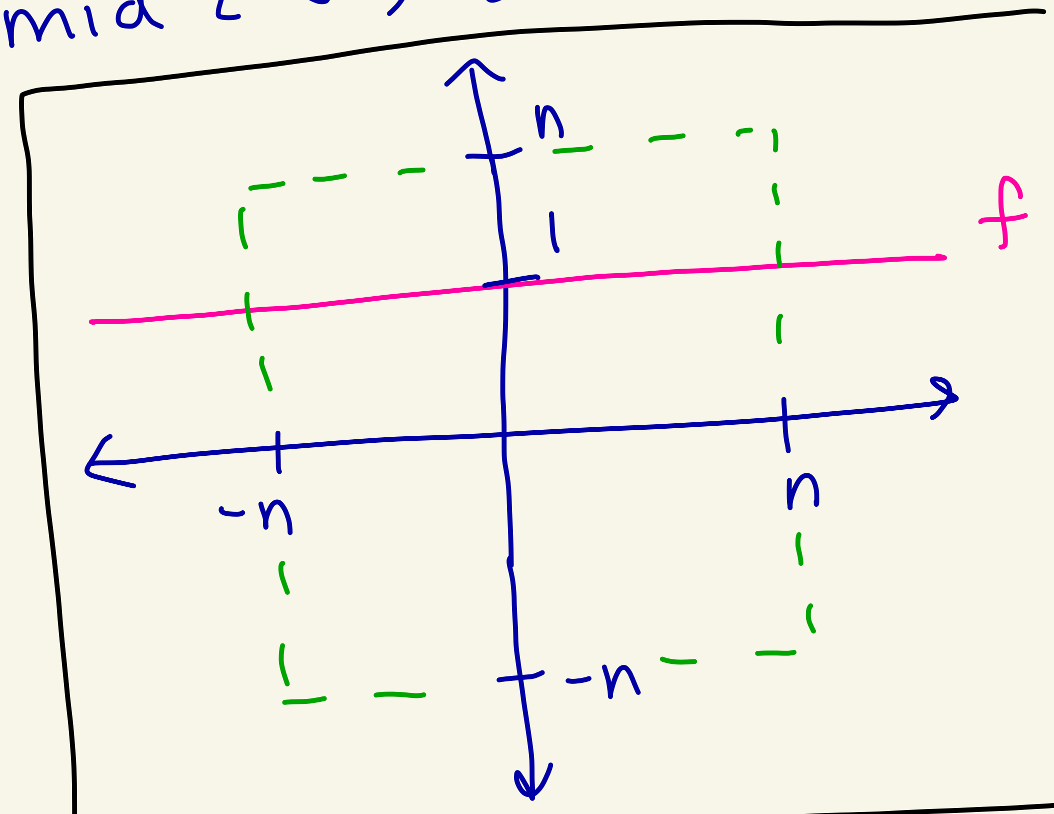
∎

Ex: Let $f = \chi_{\mathbb{R}}$.

We know $f \notin L^1$.

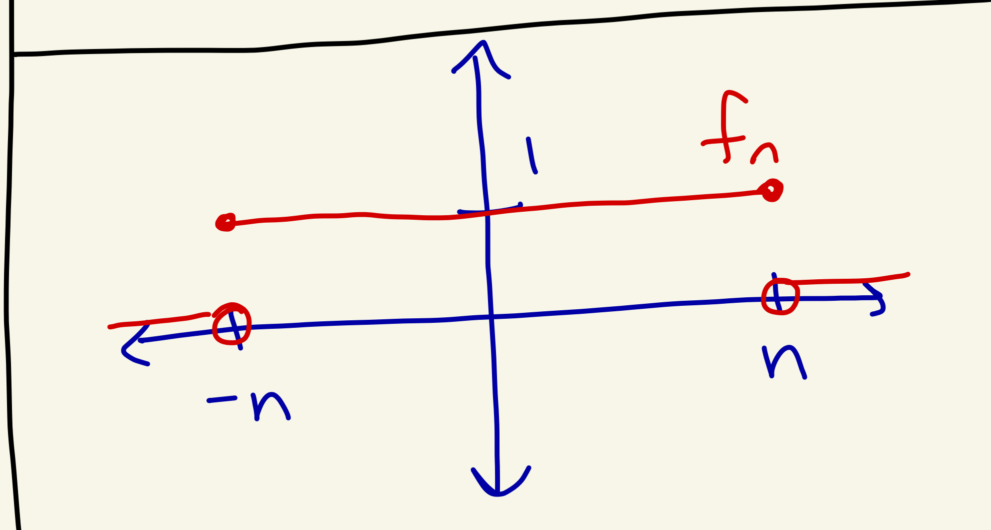
Let $g_n = n \cdot \chi_{[-n,n]}$ as in the previous theorem.

Let $f_n = \text{mid}\{-g_n, f, g_n\}$



Then,

$f_n = \chi_{[-n,n]}$



By HW 9 problem 2,

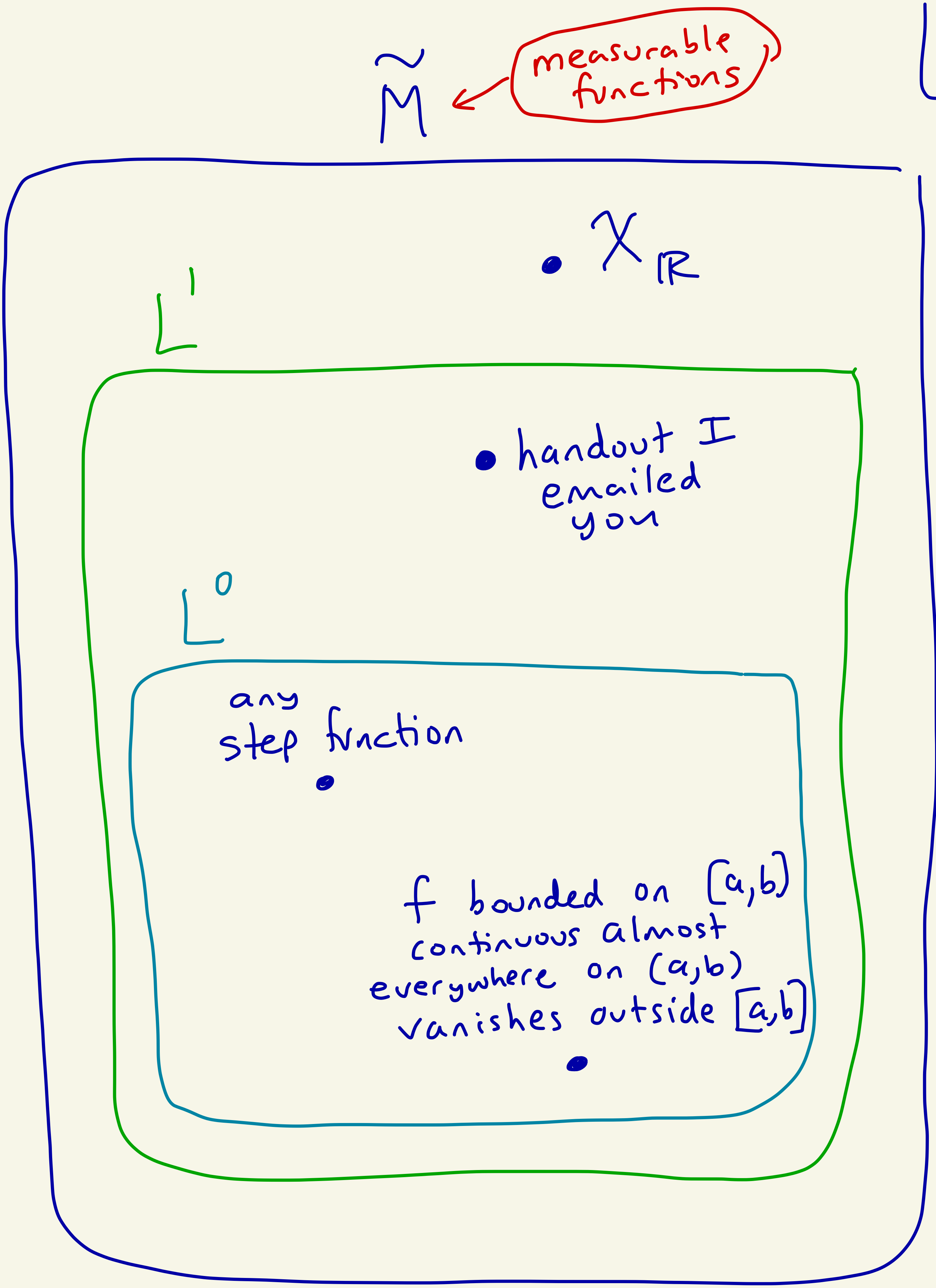
$f_n \to f$

on all of $\mathbb{R}$.

Since $f_n = \chi_{[-n,n]}$ is a step function we know $f_n \in L^1$.

So, we have a sequence $(f_n)_{n=1}^{\infty}$ of L^1 functions that converge to $f = \chi_{\mathbb{R}}$ on all of $\mathbb{R}$.

Thus, by the previous theorem, f is measurable. $\blacksquare$


$\tilde{M}$
measurable functions
$\chi_{\mathbb{R}}$
L^1
handout I emailed you
L^0
any step function
f bounded on [a,b)
continuous almost everywhere on (a,b)
vanishes outside [a,b]