Math 5800

11/8/21

Test 2

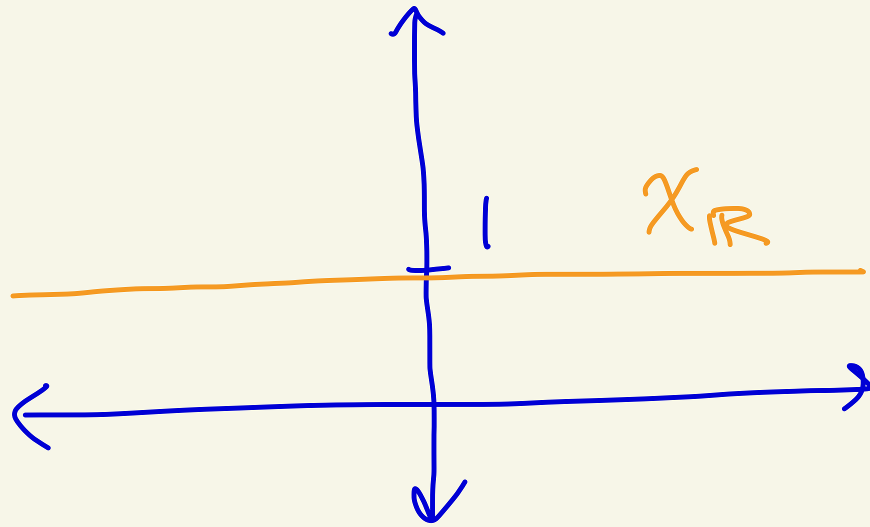
Monday Nov 15

Same as before. No class that day. Test will appear on canvas at 5am on the 15th until 12 noon on Tuesday Pick your 2.5 hour window in that time.

- I emailed the class a study guide on Friday for test 2.

I put it on the website for the class also.

Ex: $f = \chi_{\mathbb{R}}$



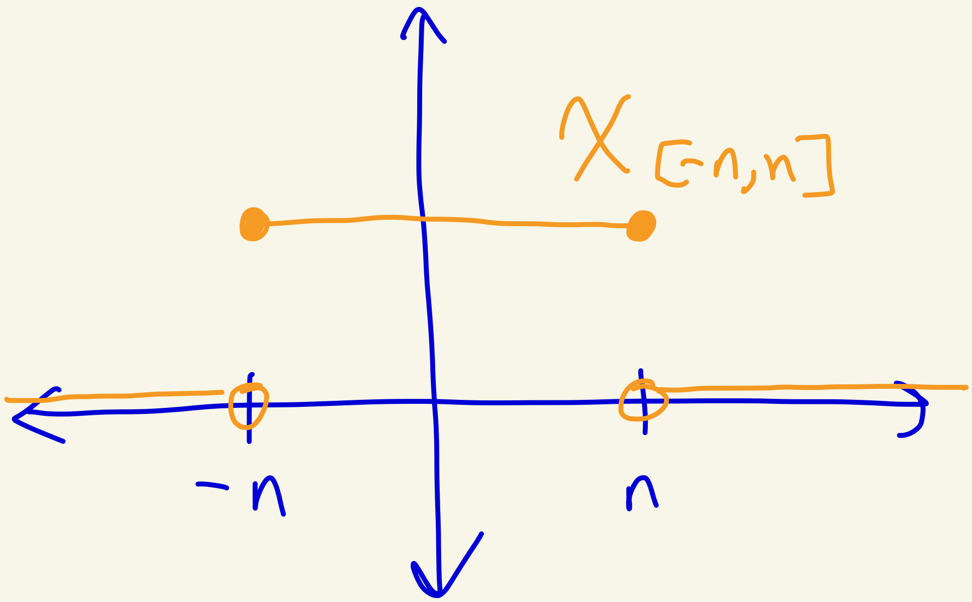
In HW you show that $\chi_{\mathbb{R}}$ is not in L^1.

We want to enlarge our space of functions to include this one and others also.

If you have $f_1, f_2, f_3, \ldots$ of L^1 functions with $f_n \to f$ almost everywhere you might not have $f \in L^1$. We need to enlarge the space to include such f.

For example, $f_n = \chi_{[-n,n]}$

then $f_n \to \chi_{\mathbb{R}}$ on all of $\mathbb{R}$.



Each $f_n = \chi_{[-n,n]}$ is in L^1

but $\chi_{\mathbb{R}}$ is not in L^1.

Def: Let $a,b,c \in \mathbb{R}$.

Define

$$\operatorname{mid}\{a,b,c\} = \begin{cases} a & \text{if } b \le a \le c \text{ or } c \le a \le b \\ b & \text{if } a \le b \le c \text{ or } c \le b \le a \\ c & \text{if } a \le c \le b \text{ or } b \le c \le a \end{cases}$$

Ex:

$\operatorname{mid}\{1, 10, -2\} = 1$ because $-2 \le 1 \le 10$

$\operatorname{mid}\{1, 5, 5\} = 5$ because $1 \le 5 \le 5$

$\operatorname{mid}\{-1, -1, -1\} = -1$ because $-1 \le -1 \le -1$

Def: Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$, $h:\mathbb{R}\to\mathbb{R}$.

Define $\text{mid}\{f,g,h\}:\mathbb{R}\to\mathbb{R}$

by

$$\text{mid}\{f,g,h\}(x) = \text{mid}\{f(x),g(x),h(x)\}$$

Ex: $f(x)=1$, $g(x)=x^2$, $h(x)=x$

$$\begin{aligned}\text{mid}\{f,g,h\}(2) &= \text{mid}\{f(2),g(2),h(2)\}\\ &= \text{mid}\{1,4,2\}\\ &= 2\end{aligned}$$

Let's draw a picture $\downarrow$

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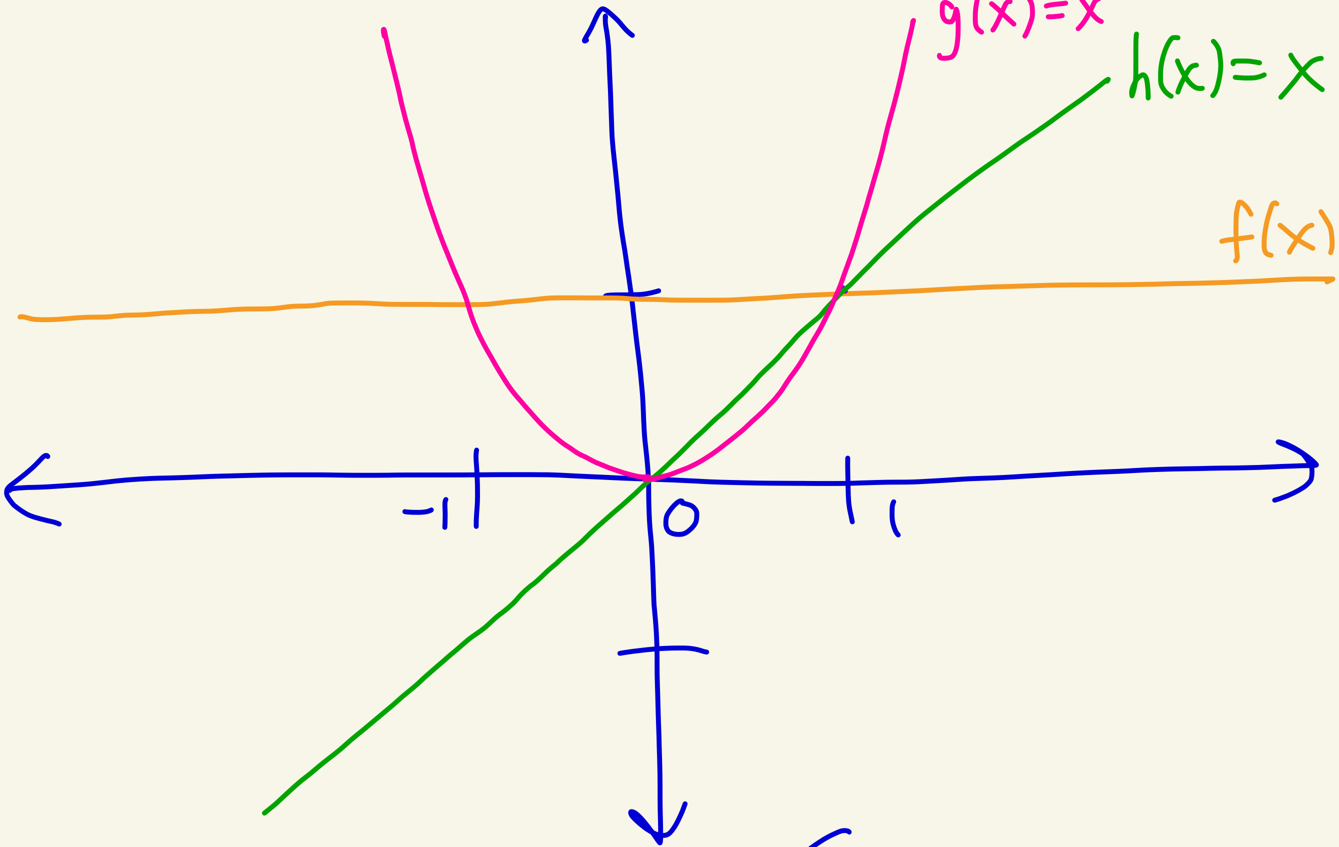
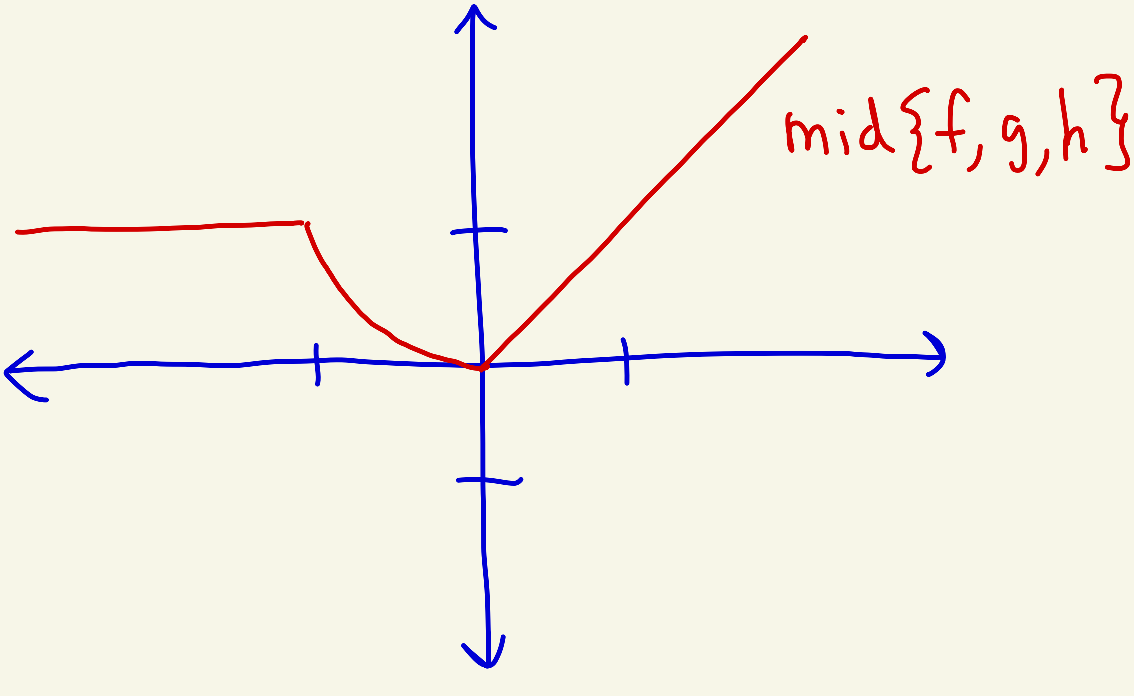


$$\operatorname{mid}\{f,g,h\}(x) = \begin{cases} 1 & \text{if } x \le -1 \\ x^2 & \text{if } -1 < x \le 0 \\ x & \text{if } 0 < x \end{cases}$$



Def: Let $a, b \in \mathbb{R}$.

Define

$$\max\{a,b\} = \begin{cases} a & \text{if } b \le a \\ b & \text{if } a \le b \end{cases}$$

and

$$\min\{a,b\} = \begin{cases} a & \text{if } a \le b \\ b & \text{if } b \le a \end{cases}$$

Ex:

$$\max\{1,5\} = 5$$

$$\min\{-1,5\} = -1$$

$$\max\{-1,-1\} = -1$$

$$\min\{-1,-1\} = -1$$

Theorem: Let $a, b \in \mathbb{R}$ and $b \geq 0$.

Then,

$$\text{mid}\{-b, a, b\} = \max\{-b, \min\{a,b\}\}$$

$$= \begin{cases} -b & \text{if } a < -b \\ a & \text{if } -b \leq a \leq b \\ b & \text{if } b < a \end{cases}$$

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Proof: Since $b \geq 0$ we have $-b \leq b$. Thus we only have three possibilities:

$$a < -b \leq b$$

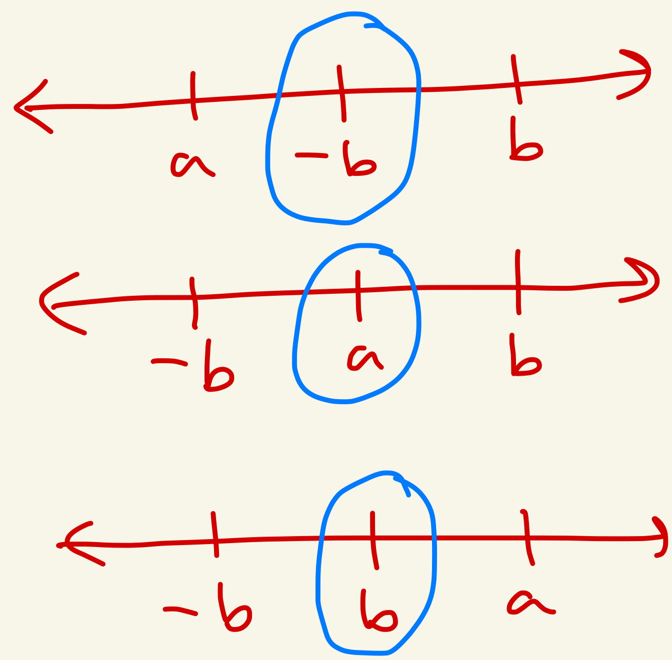
or

$$-b \leq a \leq b$$

or

$$-b \leq b < a.$$

3 cases



Thus,

$$\operatorname{mid}\{-b,a,b\} = \begin{cases} -b & \text{if } a < -b \le b \\ a & \text{if } -b \le a \le b \\ b & \text{if } -b \le b < a \end{cases}$$

$$= \begin{cases} -b & \text{if } a < -b \\ a & \text{if } -b \le a \le b \\ b & \text{if } b < a \end{cases}$$

This gives part of the result.
Now we show that

$$\operatorname{mid}\{-b,a,b\} = \max\{-b, \min\{a,b\}\}$$

Case 1: Suppose $a < -b \le b$.

Then, $\operatorname{mid}\{-b,a,b\} = -b$.
And, $\max\{-b, \min\{a,b\}\}$
$= \max\{-b, a\} = -b$

equal

<u>Case 2:</u> Suppose $-b \le a \le b$.

Then, $\text{mid}\{-b, a, b\} = a$. equal

And, $\max\{-b, \min\{a, b\}\}$

$= \max\{-b, a\} = a$

<u>Case 3:</u> Suppose $-b \le b < a$.

Then, $\text{mid}\{-b, a, b\} = b$. equal

And, $\max\{-b, \min\{a, b\}\}$

$= \max\{-b, b\} = b$

By the above 3 cases,

$\text{mid}\{-b, a, b\} = \max\{-b, \min\{a, b\}\}$

$\blacksquare$

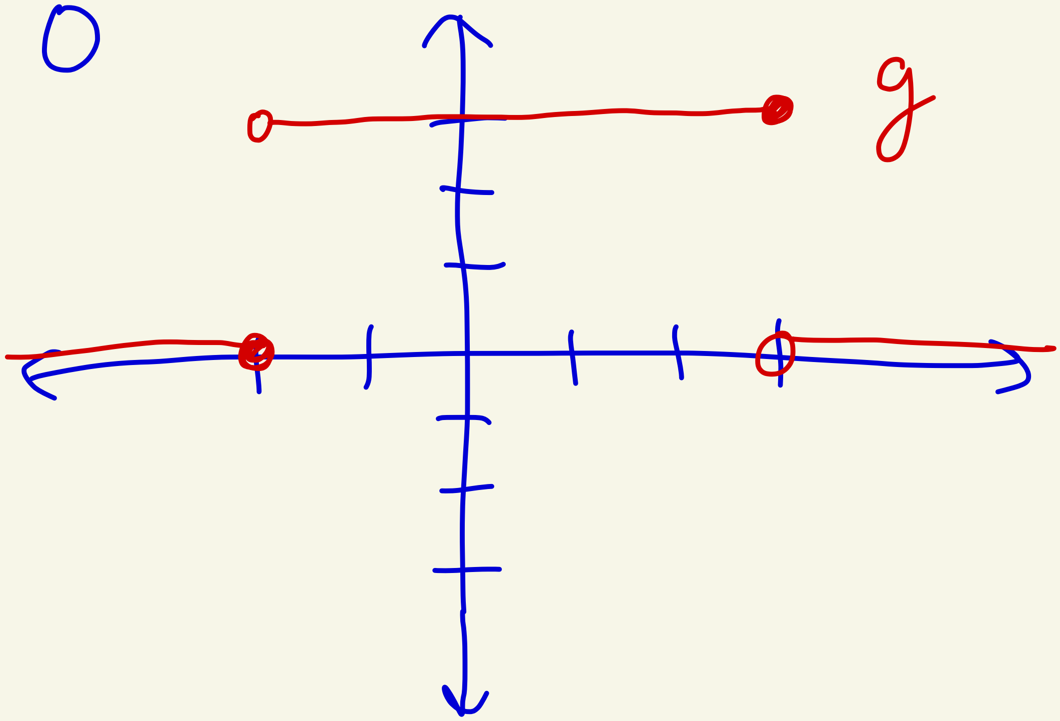
Def: Let $g : \mathbb{R} \to \mathbb{R}$.

We say that g is <u>non-negative</u> if $g(x) \geq 0$ for all $x \in \mathbb{R}$.

Shorthand notation: $g \geq 0$

Ex: $g = 3 \cdot \chi_{(-2,3]}$

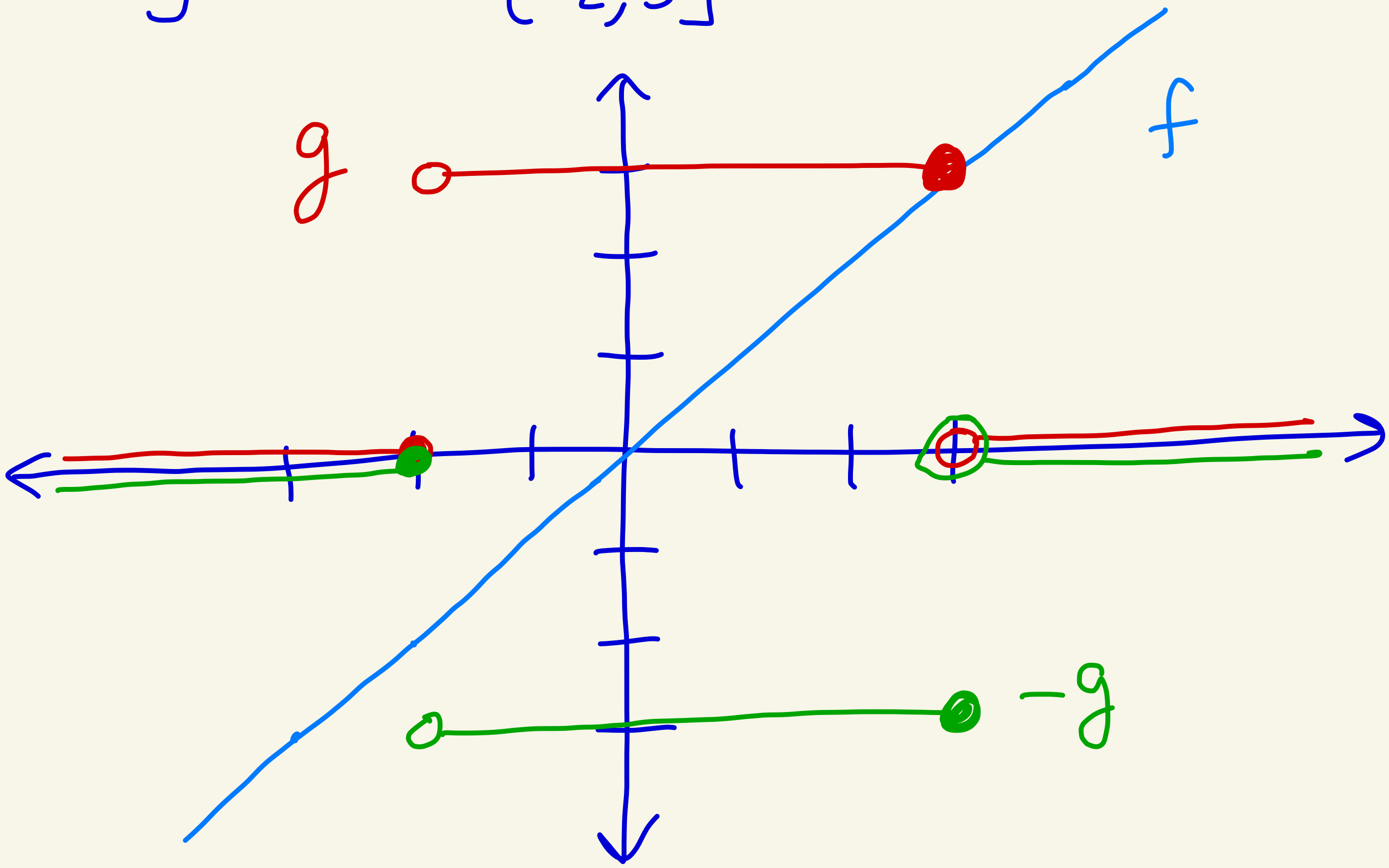
$g \geq 0$

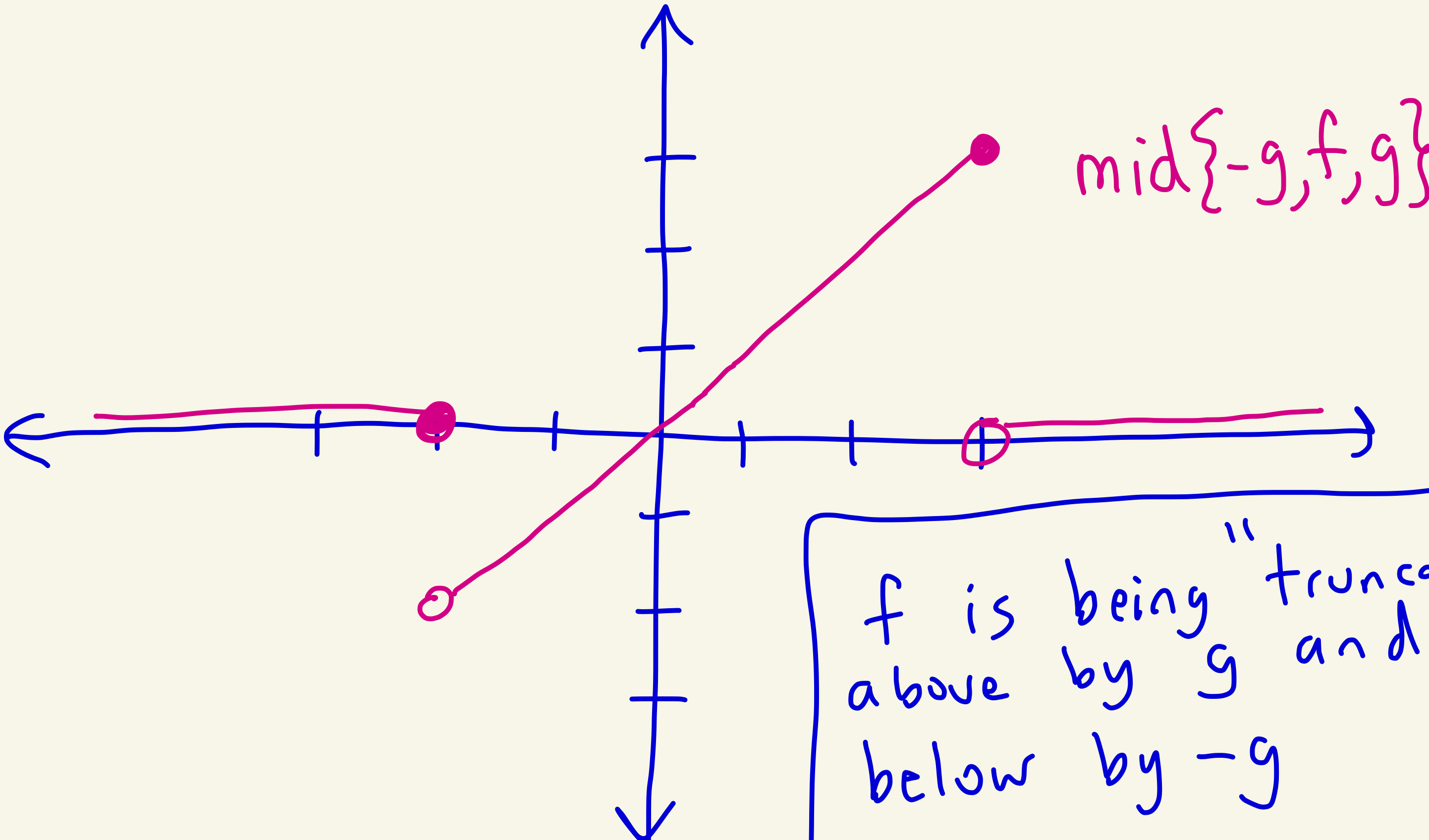


Ex: Let $f(x) = x$

and $g = 3 \cdot \chi_{(-2,3]}$

$\boxed{g \geq 0}$



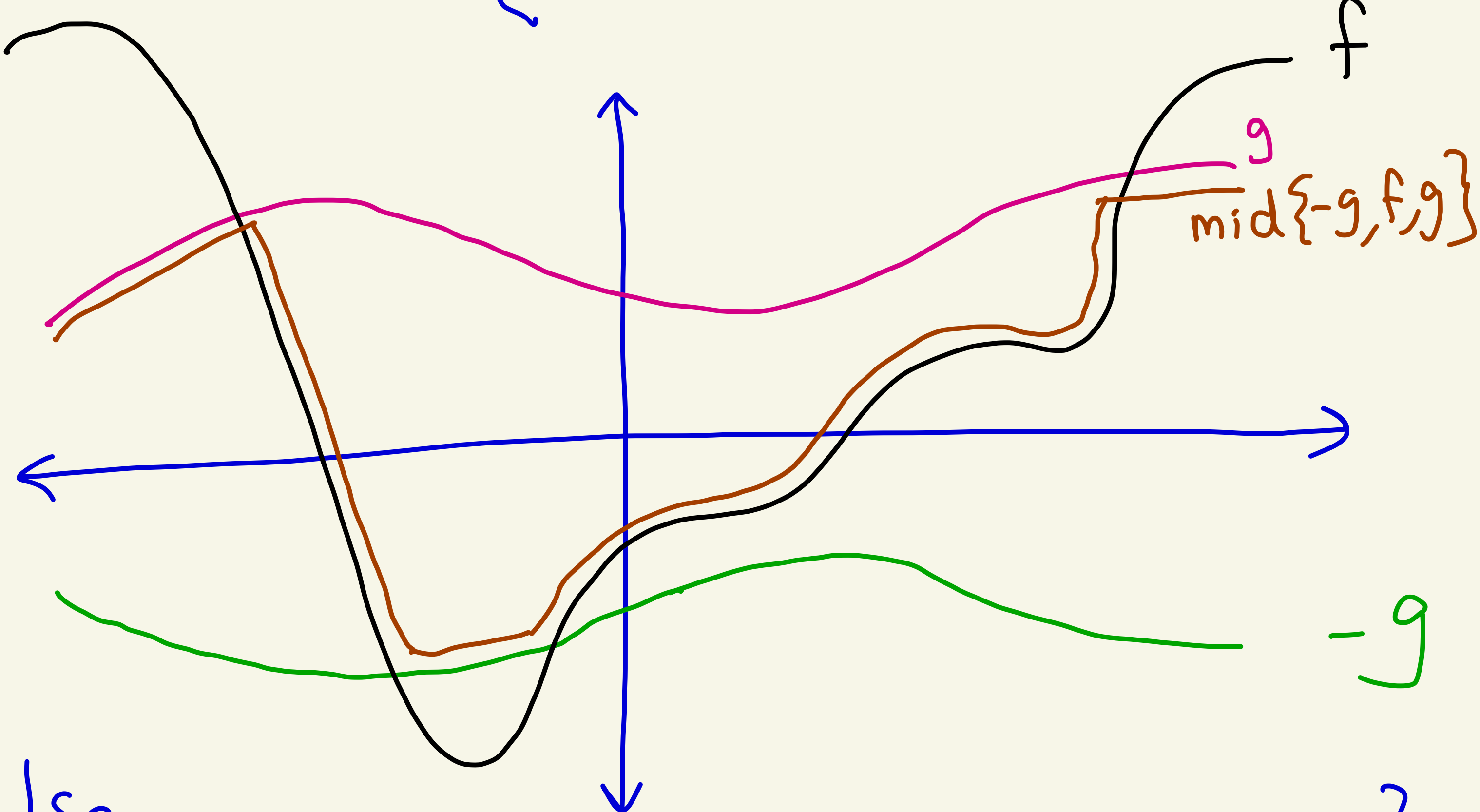


f is being "truncated" above by g and below by -g

Theorem: Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ where g is non-negative [ie $g \geq 0$]

Then,

$$\text{mid}\{-g,f,g\}(x) = \begin{cases} -g(x) & \text{if } f(x) < -g(x) \\ f(x) & \text{if } -g(x) \leq f(x) \leq g(x) \\ g(x) & \text{if } g(x) \leq f(x) \end{cases}$$



Also,

$$\text{mid}\{-g,f,g\} = \max\{-g, \min\{f,g\}\}$$

where $\max\{f,g\}(x) = \max\{f(x),g(x)\}$

and $\min\{f,g\}(x) = \min\{f(x),g(x)\}$

<u>proof</u>: Follows from previous theorem. $\boxtimes$