Math 5800

12/1/21

<u>Recall:</u> Let E be a measurable set [means χ_E is a measurable function].

Then,

$$\mu(E) = \begin{cases} \int \chi_E & \text{if } E \text{ is integrable} \\ \infty & \text{otherwise} \end{cases}$$

[means χ_E is in L^1]

<u>Proved:</u>

E has measure zero iff

E is measurable and $\mu(E) = 0$

$\boxed{\text{HW 9 \# 8}}$

Let f, g be measurable functions and $\alpha \in \mathbb{R}$. Then the following functions are all measurable

$$f + g, \quad \alpha f, \quad \min\{f, g\}, \quad \max\{f, g\}$$

$$\min\{f,g\}(x) = \min\{f(x), g(x)\}$$

$$\max\{f,g\}(x) = \max\{f(x), g(x)\}$$

<u>Theorem:</u> The set of measurable sets M forms a σ-algebra. That is,

(i) $\mathbb{R} \in M$

(ii) If $E, F \in M$, then $E \cup F \in M$ and $E - F \in M$.

(iii) If $E_1, E_2, E_3, \ldots$ is a sequence of sets in M then $\bigcup_{k=1}^{\infty} E_k \in M$

<u>Proof:</u>

(i) We already showed $\chi_{\mathbb{R}}$ is a measurable function. So, $\mathbb{R} \in M$.

(ii) Let $E, F \in M$.

Then, χ_E and χ_F are measurable functions.

Claim: $\chi_{E \cup F} = \max\{\chi_E, \chi_F\}$

pf of claim: Let $x \in \mathbb{R}$.

Suppose $x \in E \cup F$.

Then, $\chi_{E \cup F}(x) = 1$.

equal

And either $x \in E$ or $x \in F$, so either $\chi_E(x) = 1$ or $\chi_F(x) = 1$.

Thus, $\max\{\chi_E(x), \chi_F(x)\} = 1$

Suppose $x \notin E \cup F$.

Then, $\chi_{E \cup F}(x) = 0$

equal

And $x \notin E$ and $x \notin F$.

So, $\max\{\chi_E(x), \chi_F(x)\} = \max\{0, 0\}$
$= 0$

$\boxed{\text{claim}}$

Since χ_E and χ_F are both measurable functions, by HW 9 #8(d),
$\chi_{E \cup F} = \max\{\chi_E, \chi_F\}$
is also a measurable function.
Thus, $E \cup F$ is a measurable set.

What about $E - F$?
One can show that

$$\chi_{E-F} = \chi_E - \min\{\chi_E, \chi_F\}$$

Try it out.
By HW 9 #8 since χ_E, χ_F are measurable so is $\min\{\chi_E, \chi_F\}$
Thus, since χ_E and $\min\{\chi_E, \chi_F\}$ are measurable, $\chi_{E-F} = \chi_E - \min\{\chi_E, \chi_F\}$ is measurable. Thus, $E - F \in \mathcal{M}$.

(iii) Suppose $E_1, E_2, E_3, \ldots$ are measurable sets.

Let $E = \bigcup_{k=1}^{\infty} E_k$

and $S_n = \bigcup_{k=1}^{n} E_k = E_1 \cup E_2 \cup \cdots \cup E_n$

Let $f = \chi_E$ and $f_n = \chi_{S_n}$ for $n \geq 1$.

By part (ii) since $E_1, E_2, \ldots, E_n$ are measurable, so is $S_n = E_1 \cup E_2 \cup \cdots \cup E_n$.

Thus, $f_n = \chi_{S_n}$ is a measurable function for $n \geq 1$.

Claim: $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$

proof of claim: Let $x \in \mathbb{R}$.

case 1: Suppose $x \notin E = \bigcup_{k=1}^{\infty} E_k$.

Then, $x \notin S_n = \bigcup_{k=1}^{n} E_k$ for all $n \geq 1$.

Thus,

$$\lim_{n\to\infty} \underbrace{f_n(x)}_{\chi_{S_n}(x)} = \lim_{n\to\infty} 0 = 0 = \chi_E(x)$$

$$= f(x).$$

case 2: Suppose $x \in E = \bigcup_{k=1}^{\infty} E_k$.

Let $\varepsilon > 0$.

Then $x \in E_N$ for some $N \geq 1$.

Thus, $x \in S_n = \underbrace{\bigcup_{k=1}^{n} E_k}_{E_N \text{ is in here if } n \geq N}$ for all $n \geq N$

Thus if $n \geq N$ we have

$$|f_n(x) - f(x)| = |\chi_{S_n}(x) - \chi_E(x)|$$
$$= |1 - 1| = 0 < \varepsilon$$

So, $\lim_{n \to \infty} f_n(x) = f(x)$ if $x \in E$.

$\boxed{\text{Claim}}$

So, f_n is a sequence of measurable functions that converges everywhere to f.

By a theorem on Monday f is measurable.

Thus, since $f = \chi_E$ we know E is a measurable set. ∎

Theorem: If E and F are measurable sets, then $E \cap F$ is a measurable set.

proof sketch:

Show that

$$\chi_{E \cap F} = \min\{\chi_E, \chi_F\}$$

and use HW 9 #8(c) $\blacksquare$

Lemma: Suppose A and B are measurable sets and $A \subseteq B$. If B is integrable, then A is integrable.

[So, if A is not integrable, then B is not integrable]

proof:

Suppose $A, B \in \mathcal{M}$ and $A \subseteq B$.
Suppose B is integrable.
Then, $\chi_B \in L^1$.
Since $A \subseteq B$ we know
$\chi_A(x) \leq \chi_B(x)$ for all x.

HW 4 #2

Since A is a measurable set we know χ_A is a measurable function.

Note $\chi_B \geq 0$ and $\chi_B \in L^1$.

Thus, $\underbrace{\text{mid}\{-\chi_B, \chi_A, \chi_B\}}_{\text{using } \chi_A \text{ is a measurable function and } g = \chi_B} \in L^1$.

But, $-\chi_B(x) \leq \chi_A(x) \leq \chi_B(x)$

for all x.

Thus, $\text{mid}\{-\chi_B, \chi_A, \chi_B\} = \chi_A$.

So, $\chi_A \in L^1$.

Thus, A is integrable. $\blacksquare$